Numerical Analysis of Implicitly Constituted Incompressible Fluids: Mixed Formulations

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Declaration of Authorship

I declare that the contents of this thesis are, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known, nor has any part of this thesis been submitted for a degree at any other institution.

Abstract

We consider the numerical approximation of incompressible non-Newtonian flow by means of the finite element method, where the constitutive law is defined through an implicit relation $\mathbf{G}(\mathbf{S}, \mathbf{D}(\boldsymbol{u})) = \mathbf{0}$. The setting considered in this work captures models widely used in applications, such as the Bingham and the Carreau–Yasuda constitutive relations. Since in general it is not possible to solve for the shear stress $\mathbf{S}$ in the constitutive relation, the emphasis is placed on formulations treating the shear stress as a variable.

Under the assumption that the constitutive relation defines a monotone graph with r-growth, and that the finite element spaces satisfy appropriate inf-sup stability conditions, the first part of the thesis extends earlier results in the literature to provide a convergence result that guarantees that a subsequence of the numerical approximations converges weakly to a solution of the system, in the optimal range $r > \frac{2d}{d+2}$, where d is the spatial dimension. The qualitative nature of this convergence result is a consequence of the generality of the framework of implicitly constituted fluids, for which e.g. higher regularity estimates are not available. Computational examples show, nevertheless, that the numerical scheme considered exhibits the expected convergence rates, in the situations where these are available.

In the second part of the thesis we develop an augmented Lagrangian preconditioner for a stress-velocity-pressure formulation of the steady system. The preconditioner involves a specialised multigrid algorithm that makes use of a space decomposition that captures the kernel of the divergence operator, and non-standard intergrid transfer operators. Although the current theory for robust multigrid works only for symmetric and positive-definite systems (and thus does not apply to the systems considered in this thesis), the resulting preconditioner exhibits remarkable robustness properties.

In the final chapter of the thesis, the extension to the anisothermal case is carried out. We employ an implicit constitutive relation that allows for a temperature dependence of rheological parameters such as the viscosity and the yield stress, and provide convergence results for the unsteady forced convection system that takes the viscous dissipation term $\mathbf{S}:\mathbf{D}(\boldsymbol{u})$ into account, and for the steady Oberbeck–Boussinesq approximation. For the latter an augmented Lagrangian preconditioner is also introduced; this preconditioner exhibits robust convergence behaviour when applied to the Navier–Stokes and power-law systems, including temperature-dependent viscosity, heat conductivity, and viscous dissipation.

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Contents

List of Figures

Chapter 1

Introduction

In the classical theory of continuum mechanics, the balance laws of momentum, mass, and energy do not determine the behaviour of a system completely. Additional information that captures the manner in which the material to be studied responds to given stimuli is needed; this is what is commonly known as a *constitutive relation*. A constitutive relation establishes a relationship between thermodynamic fluxes and thermodynamic affinities; for instance, it can relate the stress to the deformation gradient in elasticity theory, or the heat flux and the temperature in the case of an heat conducting fluid.

If a fluid occupies part of a space represented by a connected open set $\Omega \subset \mathbb{R}^d$, where $d \in \{2,3\}$, then the evolution of the system during a given time interval $[0,T)$, for $T > 0$, is determined by the usual equations of balance of mass, momentum, angular momentum and energy, which in Eulerian coordinates take the form [MP16]:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0, \tag{1.1a}$$

$$\frac{\partial (\rho \boldsymbol{u})}{\partial t} + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) = \operatorname{div} \mathbf{T} + \rho \boldsymbol{f}, \tag{1.1b}$$

$$\mathbf{T} = \mathbf{T}^{\top}, \tag{1.1c}$$

$$\frac{\partial(\rho(e + \frac{1}{2}|\boldsymbol{u}|^2))}{\partial t} + \operatorname{div}(\rho(e + \tfrac{1}{2}|\boldsymbol{u}|^2)\boldsymbol{u}) = \operatorname{div}(\mathbf{T}\boldsymbol{u} - \boldsymbol{q}) + \rho \boldsymbol{f} \cdot \boldsymbol{u}. \tag{1.1d}$$

Here:

- $\boldsymbol{u} : [0,T) \times \overline{\Omega} \to \mathbb{R}^d$ is the velocity field;
- $\rho : [0,T) \times \overline{\Omega} \to \mathbb{R}$ is the density;
- $\mathbf{T} : (0,T) \times \overline{\Omega} \to \mathbb{R}^{d\times d}$ is the Cauchy stress;
- $e : [0,T) \times \overline{\Omega} \to \mathbb{R}$ is the internal energy;

- $\boldsymbol{q} : (0,T) \times \overline{\Omega} \to \mathbb{R}^d$ is the heat flux;
- $\boldsymbol{f} : (0,T) \times \overline{\Omega} \to \mathbb{R}^d$ is a (given) body force.

Moreover, the second law of thermodynamics requires the following inequality to be satisfied:

$$\mathsf{T} : \mathsf{D}(\boldsymbol{u}) + p_{th} \operatorname{div} \boldsymbol{u} - \frac{1}{\theta}\boldsymbol{q} \cdot \nabla\theta \geq 0, \tag{1.2}$$

where p_{th} is the thermodynamic pressure and θ is the temperature. This inequality imposes a restriction on the choice of constitutive relations.

Incompressible non-Newtonian models

This thesis will focus entirely on *incompressible* and *homogeneous* fluids; in this case $\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$ and the conservation of mass equation (1.1a) reduces to

$$\operatorname{div} \boldsymbol{u} = 0. \tag{1.3}$$

In addition, the Cauchy stress can be split in two components:

$$\mathsf{T} = -p\boldsymbol{I} + \mathsf{S}, \tag{1.4}$$

where $\boldsymbol{I}$ is the identity matrix, $p\colon (0,T)\times\Omega \to \mathbb{R}$ is the pressure (mean normal stress), and $\mathsf{S}\colon (0,T) \times \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is the shear stress (hereafter referred only as "stress"). It should be noted that a more careful derivation of this system would require one to perform e.g. a singular limit in which the speed of sound in the fluid tends to infinity [FN17], or to treat the function p as the Lagrange multiplier associated to the constraint requiring that the fluid only undergoes isochoric processes [MR05] (the interpretation of p as a Lagrange multiplier is not always appropriate, especially when dealing with non-Newtonian behaviour, see [Raj15]). In particular, what we refer to as the "pressure" in this work is not the thermodynamic pressure, which has to be specified constitutively.

One of the simplest forms of the constitutive relation between the stress and the rate of strain is given by the Newtonian constitutive relation:

$$\mathsf{S} = 2\mu\mathsf{D}, \tag{1.5}$$

where $\mu > 0$ is the (shear) viscosity of the fluid, and $\mathsf{D} = \mathsf{D}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\top})$ is the symmetric velocity gradient. This relation is linear and isotropic, and it leads to what is commonly referred to as the (incompressible) Navier–Stokes system:

$$\begin{aligned} \partial_t\boldsymbol{u} + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) - \operatorname{div}(2\nu\mathsf{D}(\boldsymbol{u})) + \nabla p &= \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{u} &= 0, \end{aligned} \tag{1.6}$$

where $\nu := \frac{\mu}{\rho_*}$ is the kinematic viscosity (ρ_* is the constant density of the fluid), and the pressure has been rescaled $p \mapsto \frac{p}{\rho_*}$. Note that the energy equation (1.1d) decouples because there is no temperature dependence in either the momentum or mass equations.

The first contributions to the mathematical theory of the incompressible Navier–Stokes system written in the modern language of functional analysis were the seminal works by Leray [Ler34] and Hopf [Hop50] (see also [KL63]), in which existence of weak solutions for large data was established. The uniqueness of a solution in the class of weak solutions remains a notoriously hard problem [CJW06].

The Newtonian constitutive relation (1.5) can be used to describe the flow of fluids like water and honey under everyday conditions [Sar10], but most of the fluids encountered in practice exhibit behaviour that cannot be captured by it; these are usually termed non-Newtonian fluids. Perhaps the simplest example of such responses is that of *shear-thinning* (sometimes called pseudoplastic) fluids, for which the viscosity can be observed to decrease with the shear-rate; some examples include ball-point pen ink, fabric conditioner, lubricating grease, blood, molten polystyrene, just to name a few [BHW89, Yas79, ABH05]. This type of response can be modeled, for instance, with the Ostwald–de Waele constitutive relation (also called the power-law model) [Ost25, dW23]:

$$\mathbf{S} = K|\mathbf{D}|^{r-2}\mathbf{D} \qquad K > 0,\ r > 1, \tag{1.7}$$

the Carreau–Yasuda relation [Car72, Yas79]:

$$\mathbf{S} = \left[\nu_\infty + (\nu_0 - \nu_\infty)(1 + \Gamma|\mathbf{D}|^2)^{\frac{r-2}{2}}\right]\mathbf{D} \qquad \nu_0, \nu_\infty \geq 0,\ \Gamma > 0,\ r > 1, \tag{1.8}$$

or that of Sisko [Sis58]:

$$\mathbf{S} = \left(\nu_\infty + \alpha|\mathbf{D}|^{r-2}\right)\mathbf{D} \qquad \nu_\infty, \alpha > 0,\ r > 1. \tag{1.9}$$

When $r > 2$, the constitutive relations (1.7)–(1.9) actually describe shear-thickening fluids (sometimes called dilatant fluids), for which the viscosity increases with shear rate. This response is less common, and can be found for example in uncooked corn starch with water (oobleck).

The relations (1.7)–(1.9) can be written in the form $\mathbf{S} = \boldsymbol{\mathcal{S}}(\mathbf{D}) := \nu(|\mathbf{D}|)\mathbf{D}$ (models of this form are sometimes called quasi-Newtonian), and they have in common that they are r-coercive, i.e. there exists a positive constant c_* such that

$$\boldsymbol{\mathcal{S}}(\mathbf{D}) : \mathbf{D} \geq c_*(|\boldsymbol{\mathcal{S}}(\mathbf{D})|^{r'} + |\mathbf{D}|^r), \tag{1.10}$$

where $r' > 1$ is the number such that $\frac{1}{r} + \frac{1}{r'} = 1$, and $\boldsymbol{\tau} : \boldsymbol{\sigma} := \mathrm{tr}(\boldsymbol{\tau}^\top \boldsymbol{\sigma})$ denotes the Frobenius inner product between two matrices $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. This condition leads to a priori estimates that suggest that the appropriate Sobolev space for the velocity field $\boldsymbol{u}$ is $W^{1,r}(\Omega)^d$.

Furthermore, the relations (1.7)–(1.9) are also *monotone*, that is, for any $\mathbf{D}_1, \mathbf{D}_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, the following condition holds

$$(\boldsymbol{\mathcal{S}}(\mathbf{D}_1) - \boldsymbol{\mathcal{S}}(\mathbf{D}_2)) : (\mathbf{D}_1 - \mathbf{D}_2) \geq 0. \tag{1.11}$$

The mathematical analysis of systems with the structure mentioned above was pioneered by Ladyzhenskaya [Lad69] and Lions [Lio69]. Their existence proof is based on methods of monotone operator theory, and assumes that $r \geq \frac{3d+2}{d+2}$ ($r \geq \frac{3d}{d+2}$ in the stationary case); this restriction ensures that the velocity $\boldsymbol{u}$ is an admissible test function in the weak formulation, which implies that there is an energy identity available, and so Minty's trick can be applied to identify the nonlinear limit in the constitutive relation.

The results of Lions and Ladyzhenskaya were extended to the range $r > \frac{2(d+1)}{d+2}$ ($r > \frac{2d}{d+1}$ in the steady case), by employing an L^∞-truncation technique in the works [FMS97, Růž97, Wol07]. The application of a Lipschitz truncation made it possible to extend further the existence result to the range $r > \frac{2d}{d+2}$ [FMS03, DMS08, DRW10, BDF12, BDS13] (see also related results in [MNRR96, MNR93, BBN94, Ama94, MNR01, BdV05, DR05, BP07, BKR11]). This can be considered the optimal range because it guarantees that $W^{1,r}(\Omega)^d$ compactly embeds into $L^2(\Omega)^d$, and thus the convective term can be handled as a compact perturbation.

Another type of non-Newtonian response is that of fluids with a *yield stress* (also called viscoplastic fluids). These are materials which flow only when the magnitude of the shear stress $\mathbf{S}$ exceeds some critical value, and otherwise behave like a solid; this is usually expressed by means of the dichotomy:

$$\begin{cases} |\mathbf{S}| \leq \tau_* \iff & \mathbf{D} = \mathbf{0}, \\ |\mathbf{S}| > \tau_* \iff & \mathbf{S} = 2\nu(|\mathbf{D}|)\mathbf{D} + \dfrac{\tau_*}{|\mathbf{D}|}\mathbf{D}, \end{cases} \tag{1.12}$$

where $\tau_* \geq 0$ is the yield stress, and ν is a function defining for example any of the relations (1.7)–(1.9). This is usually called the Herschel–Bulkley constitutive relation [HB26], and when $\nu(|\mathbf{D}|) \equiv \nu_0$ is a constant then it is called the Bingham constitutive relation [Bin22]. Such models can be used to describe waxy crude oil, toothpaste, paint, pastes, drilling muds, and mango jam, among other things [BDY83, GW10, BS13, LF09] (see [BFO14] for a nice survey on various aspects of the modeling and simulation of viscoplastic fluids).

There exist mainly two approaches in the mathematical analysis of viscoplastic flow: the use of variational methods by framing the problem in terms of variational inequalities [DL76, FG83, Lad68, NW79, Ser91, LS95, FS00], or by enforcing the constitutive relation pointwise and approximating it with regularised constitutive laws [She02, MRS05, ER12]; with the exception of [ER12], all the results are restricted to the case $r > \frac{3d+2}{d+2}$ ($r > \frac{3d}{d+2}$ in the stationary case). While it is possible to study (1.12) as a model of the form $\mathbf{S} = \boldsymbol{\mathcal{S}}(\mathbf{D})$, this forces one to consider it as a discontinuous or multi-valued constitutive relation (this is the point of view taken in [GMŚ07, Mam07]); an important observation is that (1.12) can be very naturally written as a *continuous implicit relation* instead:

$$2\nu(|\mathbf{D}|)\mathbf{D} = \frac{(|\mathbf{S}| - \tau_*)^+}{|\mathbf{S}|}\mathbf{S}, \tag{1.13}$$

where it is assumed that the expression on the right hand side takes the value $\mathbf{0}$ when $\mathbf{S} = \mathbf{0}$. This motivates the introduction of a general constitutive relation of the form

$$\mathbf{G}(\cdot, \mathbf{S}, \mathbf{D}(\boldsymbol{u})) = \mathbf{0} \qquad \text{a.e. in } Q, \tag{1.14}$$

where $\mathbf{G}\colon Q \times \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^{d\times d}_{\text{sym}}$; here Q is used to denote the parabolic cylinder $(0,T)\times\Omega$. The relationship between $\mathbf{S}$ and $\mathbf{D}$ is not necessarily explicit anymore, and the equation (1.14) is even more general in that it allows the constitutive relation to vary with space and time. The framework of implicitly constituted fluids was introduced by Rajagopal in [Raj03, Raj06]; this provided an alternative approach to the phenomenological theory of constitutive relations that enabled the description of complex rheological behaviour not achievable by the traditional models, and has been very fruitful in the modeling of non-Newtonian fluids (see e.g. [MPR10, PR12, LRR13, PP15, JP15] for further developments). A very important feature of implicitly constituted models is that they are compatible with the thermodynamical framework of Rajagopal and Srinivasa [RS00, RS04, RS08] that guarantees the consistency of the constitutive relations with the second law of thermodynamics. This procedure consists of specifying constitutive relations for the Helmholtz free energy and the rate of entropy production (two scalar quantities), and using that information to derive a relation between the tensorial quantities $\mathbf{S}$ and $\mathbf{D}$ in a way that the thermodynamical consistency is guaranteed (as opposed to trying to verify it a posteriori). This method has been used to obtain completely new constitutive relations, e.g. to model the vulcanization of rubber [KR11], or the response of asphalt binder [MRT15].

The framework defined by (1.14) can also incorporate very naturally models in which the viscosity depends on the shear stress $\mathbf{S}$ (this type of response can be observed in ice, poly(vinyl choride) solutions and molten polyethylene, for instance [MB65, SMP94, Gle55, See64, Bla95, PW03]), such as the Ellis constitutive relation (see e.g [MB65]):

$$\mathbf{S} = \frac{\nu_0}{1 + \alpha|\mathbf{S}|^{q-2}}\mathbf{D} \qquad \nu_0, \alpha > 0,\ q \in (1,2), \tag{1.15}$$

or the model introduced by Glen [Gle55]:

$$\mathbf{S} = \alpha|\mathbf{S}|^{q-2}\mathbf{D} \qquad \alpha > 0,\ q > 1. \tag{1.16}$$

The symmetric treatment of $\mathbf{S}$ and $\mathbf{D}$ by models of the type (1.14) also offers the possibility of swapping the roles of $\mathbf{S}$ and $\mathbf{D}$ in models with an activation parameter such as (1.12). This leads to constitutive relations describing an inviscid Euler fluid before activation ($\mathbf{S} = \mathbf{0}$), and a power-law or Navier–Stokes fluid when the magnitude of the symmetric velocity gradient surpasses some critical value. Models of this type (sometimes called of activated Euler type) had not been studied prior to the introduction of implicitly constituted models, due to the prevalence of explicit relations of the kind $\mathbf{S} = \boldsymbol{\mathcal{S}}(\mathbf{D})$ in the literature, and merit further consideration (see [BMR20] for a classification of various models defined by implicit constitutive relations); one possible application could be the simulation of boundary layers, where the basic assumption is that the effects of viscosity are negligible away from solid walls.

Although they will not be considered in this work, models with a pressure dependent viscosity and/or yield stress are in fact more naturally written as an implicitly constituted relation [BMR09, HLS12, JFP06]:

$$\mathbf{T} = -p\mathbf{I} + \left(2\nu(|\mathbf{D}|, p) + \frac{\tau(p)}{|\mathbf{D}|}\right)\mathbf{D} = \left(2\nu(|\mathbf{D}|, \tfrac{1}{d}\operatorname{tr}(\mathbf{T})) + \frac{\tau(\frac{1}{d}\operatorname{tr}(\mathbf{T}))}{|\mathbf{D}|}\right)\mathbf{D}. \tag{1.17}$$

Such models would not be entirely justified with the traditional approach, since in classical continuum mechanics it is assumed that the constraint forces do not perform work, and therefore the constitutively determined part cannot depend on the pressure [Raj03]. Implicitly constituted theories are also advantageous in the modeling of viscoelastic fluids [Raj03], fluids that exhibit a non-monotone response between $\mathbf{S}$ and $\mathbf{D}$ [JP18, JMPT19], and elastic solids, including models with nonlinear behaviour at small strains [Raj07, Raj10, Raj14, BMRS14].

For a rigorous mathematical analysis of models of implicitly constituted fluids of the type (1.14), the reader is referred to [BGMŚ09, BGMŚ12]. Existence of weak solutions for problems of this type was obtained in [BGMŚ09] and [BGMŚ12] for the steady and unsteady cases, respectively. Some extensions include [BMPS18, MZ18, PR18, BM16], where additional physical responses are incorporated into the system.

Boundary conditions

An important issue in the modeling of fluid flow is the choice of boundary conditions. In this work, the analysis will consider internal flows only, that is, we will assume that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, where $\boldsymbol{n}$ denotes the outward pointing normal vector to Ω. Let us denote the tangential component of the velocity by $\boldsymbol{u}_\tau := \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n}$. In the mathematical modeling of viscous fluids the most widespread choice is the *no-slip* condition:

$$\boldsymbol{u}_\tau = 0, \tag{1.18}$$

which assumes that the fluid sticks to solid walls. An important issue that arises as a consequence of this choice is that, for unsteady flow in Lipschitz domains (such as the polyhedral domains used commonly with the finite element method), even for Newtonian flow, the pressure is in general only a distribution in time [Tem84]: $p \in W^{-1,\infty}(0,T;L^2(\Omega))$, which can cause technical difficulties.

In some applications it has been observed that the no-slip condition (1.18) does not provide an adequate approximation to experimental results: in certain regimes fluids exhibit some degree of slip at the walls [HL03, Den01, Den04]. Navier's boundary condition is perhaps the simplest boundary condition that allows for slip [Nav23]:

$$\boldsymbol{u}_\tau = -\gamma(\mathbf{S}\boldsymbol{n})_\tau \qquad \gamma > 0. \tag{1.19}$$

Besides it being potentially a better model, the advantage of Navier's boundary condition is that (under appropriate conditions), it allows one to obtain a globally integrable pressure $p \in L^1((0,T) \times \Omega)$. Boundary conditions can themselves be considered as constitutive relations and one should ensure that they are thermodynamically consistent [MP16]. Further generalisations have appeared in the literature in which the boundary condition can even be defined implicitly $\boldsymbol{h}(\boldsymbol{u}_\tau, (\mathbf{S}\boldsymbol{n})_\tau) = \boldsymbol{0}$ (see e.g. [BM17, BMR20]), but these will not be considered in this thesis.

The temperature equation

So far we have mainly discussed models that neglect thermal effects, but in applications they can be of fundamental importance (e.g. the viscosity of most liquids decreases with temperature) [Bri31, SW59, CSC+78, YPS81, SDL85, Sze98, AHMM03].

In this work we will assume that the internal energy and the temperature are related through the linear equation $e = c_v\theta$, where $c_v > 0$ is the specific heat capacity at constant volume. As a constitutive relation we will exclusively work with Fourier's law:

$$\boldsymbol{q} = -\kappa(\theta)\nabla\theta, \tag{1.20}$$

where $\kappa > 0$ is the heat conductivity and it possibly depends on the temperature. If we multiply the momentum equation (1.1b) by $\boldsymbol{u}$ and substract the resulting equation from the energy balance (1.1d), we obtain a form of the temperature equation that is more commonly employed in applications:

$$\partial_t(\rho_* c_v\theta) + \operatorname{div}(\rho_* c_v\theta\boldsymbol{u}) - \operatorname{div}(\kappa(\theta)\nabla\theta) = \mathbf{S}:\mathbf{D}(\boldsymbol{u}). \tag{1.21}$$

When dealing with weak solutions, the equations (1.21) and (1.1d) are in general no longer equivalent, as the argument requires one to test the momentum equation with $\boldsymbol{u}$. The effects of viscous dissipation are very often neglected and the equation is written as

$$\partial_t\theta + \boldsymbol{u}\cdot\nabla\theta - \operatorname{div}(\tilde{\kappa}(\theta)\nabla\theta) = 0, \tag{1.22}$$

where $\tilde{\kappa} := \frac{\kappa}{c_v\rho_*}$ is the thermal diffusion rate. The effects of viscous heating have been observed in many applications to be non-negligible (see e.g. [HMW75, THTS74, VPC76, Ost58]), and for the most part we will give a preference to the system (1.21) over (1.22).

When the viscous dissipation term $\mathbf{S}:\mathbf{D}(\boldsymbol{u})$ is neglected, besides a possible temperature dependence in the rheological parameters, the momentum and temperature equations are only coupled through the advection term $\boldsymbol{u}\cdot\nabla\theta$, leading to what is usually termed *forced convection*: heat is transported mainly thanks to the fluid motion. An additional heat transport mechanism that is often considered is that of *natural convection*, for which the main assumption is that the body force is given by gravity and that the variation of density depends linearly on the temperature perturbations:

$$\begin{aligned} \boldsymbol{f} &= -g\boldsymbol{e}_d, \\ \rho - \rho_* &= \rho_*\beta(\theta - \theta_C), \end{aligned} \tag{1.23}$$

where g is the acceleration due to gravity, $\boldsymbol{e}_d$ is the unit vector pointing against gravity, β is the volumetric coefficient of thermal expansion, and ρ_* and θ_C are reference values for the density and temperature, respectively. This is called the Oberbeck–Boussinesq approximation [Obe79, Bou03] and leads to the system:

$$\rho_*\partial_t \boldsymbol{u} + \rho_* \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) - \operatorname{div}\mathbf{S} + \nabla p = -\rho_*\beta g(\theta-\theta_C)\boldsymbol{e}_d \quad \text{in } \Omega, \tag{1.24a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.24b}$$

$$\rho_* c_p(\partial\theta + \boldsymbol{u}\cdot\nabla\theta) - \operatorname{div}(\kappa(\theta)\nabla\theta) = 0 \quad \text{in } \Omega, \tag{1.24c}$$

where c_p is the specific heat capacity at constant pressure. Strictly speaking, a rigorous derivation of the system (1.24) involves performing a singular limit in which one assumes that the fluid is mechanically incompressible but thermally compressible [GG76, RRS96, FN09, KR16]. Viscous dissipation can also be taken into account by the Oberbeck–Boussinesq approximation, and the following equation would be used instead of (1.24c) (see [KRT00, KRT06] for a rigorous derivation):

$$\rho_* c_p \partial_t\theta + \rho_* c_p \operatorname{div}(\boldsymbol{u}\theta) + \beta\rho_* g\theta\boldsymbol{u}\cdot\boldsymbol{e}_d - \operatorname{div}(\kappa(\theta)\nabla\theta) = \mathbf{S}:\mathbf{D}(\boldsymbol{u}). \tag{1.24d}$$

The addition of the adiabatic heating term $\beta\rho_* g\theta\boldsymbol{u}\cdot\boldsymbol{e}_d$ is necessary to ensure the energy is appropriately balanced [BLP92, THTS74].

The incompressible system with (1.21) and the Oberbeck–Boussinesq system (1.24) have different origins; e.g. the temperature appearing in (1.21) is an absolute temperature (and is therefore strictly positive), while the one appearing in (1.24) is a perturbation with respect to a reference temperature. However, both systems have a similar enough structure so that the same tools can be used to analyse them.

Several works have been published that tackle the question of existence of solutions for systems describing incompressible heat-conducting fluids, but some simplifying assumptions were usually made: viscous dissipation was ignored in [Mor88, Kag93, Gon90, His91, HY92, MRT94, DG98, SK00, FN09], the works [Con97, CM97, Con00, MRT94] employed a setting in which the velocity $\boldsymbol{u}$ is an admissible test function in the balance of momentum (which excludes the Navier–Stokes model in three dimensions), and a weaker notion of solution such as a distributional solution or a weak solution with a defect measure was considered in [Lio96, DG98, Rou01, NR01, ABG09] (see also [Ama95, Ben11] for existence results for strong solutions with small data). The main difficulty in the analysis is that the viscous heating term $\mathbf{S}:\mathbf{D}(\boldsymbol{u})$ is a priori only in L^1, which makes the application of compactness arguments problematic.

A breakthrough came with the work of Bulíček, Feireisl, and Málek [BFM09] (see also [FM06]), where it was observed that, even though it contains additional couplings, the equation for the total energy (1.1d) is more amenable to weak convergence arguments than (1.21) and therefore should be preferred in the analysis; in particular, the existence of bona fide weak solutions for the Navier–Stokes–Fourier system with temperature-dependent viscosity and heat-conductivity was established for large data. The relation (1.21) becomes an inequality:

$$\partial_t(\rho_* c_v \theta) + \operatorname{div}(\rho_* c_v \theta \boldsymbol{u}) - \operatorname{div}(\kappa(\theta)\nabla\theta) \geq \mathbf{S} : \mathbf{D}(\boldsymbol{u}), \tag{1.25}$$

which acts as an entropy inequality and provides a generalisation of the concept of suitable weak solutions of the isothermal Navier–Stokes system of Caffarelli, Kohn and Nirenberg [CKN82]. This existence result was extended by Maringová and Žabenský to the implicitly constituted setting [MZ18], employing a constitutive relation that in particular captured the Bingham and activated Euler models with temperature-dependent viscosity and activation parameters. A drawback of this approach is that the equation for the total energy (1.1d) contains the pressure p, and thus boundary conditions that allow some slip such as (1.19) must be employed, in order to guarantee its global integrability.

Numerical approximation of non-Newtonian flow

The past couple of decades have seen many contributions to the numerical analysis of systems with an r-structure (recall (1.10)). Usually employing tools firstly developed for the study of elliptic and parabolic p-Laplace-type systems [GM75, Cho89, BL93a, BL94a, Far98, DER07, BR20a, BR20b], the first results dealing with non-Newtonian flow neglected the convective term and focused mostly on a priori [BN90, DG90, BL93b, San93, BL94b, Bao02, BBDR12, Hir13, ER18] and a posteriori [BEA91, Sim95, Pad97, BB98, CF01, BS08] error estimates, often focusing on Carreau's constitutive relation. Of special importance we mention the works of Hirn and Belenki et al. [BBDR12, HLS12], where (motivated by ideas from Barrett and Liu [BL94b]), a nonlinear measure of the error (the so-called quasi-norms) allowed the authors to establish optimal error estimates for $r \leq 2$. This argument was then extended to the unsteady case in [ER18]. For a different approach for the quasi-Stokes system with singular forcing see also the recent work [OS20].

The analysis of the unsteady systems with an r-structure that takes into account the convective term began with the work of Diening, Prohl and Růžička [PR01, DPR02] (see also the important contributions of Heywood and Rannacher to the analysis of the Newtonian problem [HR82, HR86, HR88, HR90]). In those works, suboptimal error estimates for some interval $r \in (r_0, 2]$, with $r_0 < 2$, and a fully implicit discretisation are obtained; these estimates yield a convergence result to the strong solution that is known to exist, at least for short times, in the setting therein considered (the use of periodic boundary conditions is crucial). By adding a stabilisation term and using a semi-implicit discretisation, this result was extended to the range $r \in (\frac{3}{2}, 2]$ in [DPR06]; the use of quasi-norms then allowed the authors to avoid the stabilisation terms and to obtain an optimal estimate with respect to time for the first time in [BDR09]. Finally, by combining ideas from [BDR09, BBDR12, DER07], an optimal estimate with respect to both space and time was established in [BDR15].

The approach just described based on error estimates is likely to be of little use in the analysis of implicitly constituted fluids, since there are no higher regularity estimates available. For this reason, in this work we will aim for qualitative convergence results that establish the weak convergence of (a subsequence of) the sequence of numerical approximations to a weak solution of the system. This point of view is taken in [CHP10], where convergence for the system with an explicit constitutive relation with r-structure and no-slip boundary conditions is established for $r > 2\frac{d+1}{d+2}$, with the help of an L^∞-truncation; see also [Emm08] for a convergence result related to a two-step time discretisation for $r \geq \frac{3d+2}{d+2}$.

Regarding the numerical analysis of implicitly constituted fluids, very few results have been published so far. In [DKS13] the convergence of a finite element discretisation to a weak solution of the problem was proved for the steady case, and the corresponding a-posteriori analysis was carried out in [KS16]. The analysis in [DKS13] follows that of [BFM09], where in particular the nonlinear limit associated with the constitutive relation is identified using Young measures; the development of a discrete version of the Lipschitz truncation in that work made it possible to obtain a convergence result in the range $r > \frac{2d}{d+1}$ for discretely divergence-free elements, and $r > \frac{2d}{d+2}$ for exactly divergence-free elements. More recently, this approach was extended to the time-dependent case in [ST19], where thanks to the addition of a penalty term, convergence of a fully implicit discretization was established in the optimal range $r > \frac{2d}{d+2}$. Also, several finite element discretisations were compared computationally in [HMST17] for problems with Bingham and stress-power-law-like rheology.

Numerical methods for the incompressible Navier–Stokes equations are usually based on a velocity-pressure formulation, and extensive studies have been carried out over the years in relation to this (see e.g. [GR86, BF91]). Such a formulation is possible, because in the case of a Newtonian fluid the explicit constitutive relation $\mathbf{S} = 2\nu\mathbf{D}(\boldsymbol{u})$ allows one to eliminate the shear stress $\mathbf{S}$ from the momentum equation. In contrast, formulations that treat the stress as a fundamental unknown have also been introduced to study problems in elasticity and incompressible flows [ABD84, BFT93,

FF93, FM96, AW02, EHS08, FNP08, FNP09, How09, HW13]; the key advantages of these formulations are that they are naturally applicable to nonlinear constitutive models where it is not possible to eliminate the stress, and that they allow the direct computation of the stress without resorting to numerical differentiation. In this work we will consider the mathematical analysis of mixed formulations that treat the stress as an unknown, and illustrate their performance by means of numerical simulations.

Knowing that a convergent discretisation is available, a natural question that follows is how to solve the resulting discrete systems in an efficient manner. After Newton linearisation the system takes the following form

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ p \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ g \end{bmatrix}, \tag{1.26}$$

where $\boldsymbol{z} = (\mathbf{S}, \boldsymbol{u})$, $\boldsymbol{z} = (\theta, \boldsymbol{u})^\top$, or $\boldsymbol{z} = (\mathbf{S}, \theta, \boldsymbol{u})^\top$, depending on whether a 3-field or a 4-field formulation is employed, and whether one is dealing with the isothermal or anisothermal system; B represents the divergence operator acting on the velocity space. After performing Gaussian elimination on the blocks, the problem of solving (1.26) reduces to solving smaller systems involving A and the Schur complement $S := -BA^{-1}B^\top$. In many cases, such as in a velocity-pressure formulation of the Stokes system, A represents a symmetric and coercive operator which can be inverted efficiently, and so the challenge is to develop an effective and efficient approximation for the Schur complement inverse $\tilde{S}^{-1}$. For the Stokes system with constant viscosity ν it is known that the choice $\tilde{S}^{-1} = -\nu M_p^{-1}$, where M_p is the pressure mass matrix, results in a spectrally equivalent preconditioner [SW94, MW11]. When the convective term is introduced to the formulation, the performance of this strategy degrades as the Reynolds number Re gets larger (meaning that the number of Krylov subspace iterations per nonlinear iteration grows with Re) [EHS+06]. This loss of robustness occurs also with other well-known preconditioners, such as the PCD [KLW02] and LSC [EHS+06] preconditioners (see e.g. [ESW14]). Block preconditioners based on PCD for the steady version of the Newtonian Oberbeck–Boussinesq system (1.24) without viscous dissipation were proposed in [HK12, KABH17], where it was also observed that the number of linear iterations increased strongly with the Rayleigh number Ra.

Alternatively, one can consider the system with an augmented Lagrangian term, with $\gamma > 0$:

$$\begin{bmatrix} A + \gamma B^\top M_p^{-1} B & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ p \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} + \gamma B^\top M_p^{-1} g \\ g \end{bmatrix}, \tag{1.27}$$

which has the same solution as (1.26), since $B\boldsymbol{z} = g$. The advantage of this is that using the Sherman–Morrison–Woodbury formula (see e.g. [Bac06]), the Schur complement can be approximated in a straightforward way:

$$\begin{aligned} S^{-1} &= (-B(A + \gamma B^{\top} M_p^{-1} B)^{-1} B^{\top})^{-1} = -(BA^{-1}B^{\top})^{-1} - \gamma M_p^{-1} \\ &\approx -(\nu + \gamma) M_p^{-1}, \end{aligned}$$

and the approximation gets better as $\gamma \to \infty$. The difficulty now becomes solving the linear system associated with top block $A+\gamma B^{\top} M_p^{-1} B$ efficiently, since the augmented Lagrangian term possesses a large kernel (the set of all discretely divergence-free velocities). This approach was used for the 2D Navier–Stokes system by Benzi and Olshanskii [BO06] and later extended to three dimensions by Farrell, Mitchell and Wechsung [FMW19]. The strategy for efficiently solving the top block in these works was based on ideas developed by Schöberl in the context of nearly incompressible elasticity [Sch99b, Sch99a], where it became clear that constructing robust relaxation and transfer operators is essential for obtaining a γ-robust multigrid algorithm.

Aim and Outline

The convergence results here could be considered an extension of the works [DKS13, ST19, HMST17]. One of the advantages of the approach presented here with respect to [DKS13, ST19] is that it can handle the constitutive relation in a more natural way, since the stress plays a more prominent role in the weak formulation considered. In addition, in [DKS13, ST19] no numerical simulations were presented. On the other hand, while extensive numerical computations with 3-field and 4-field formulations were performed in [HMST17], no convergence analysis of the methods considered was discussed. The work presented here fills this gap.

After introducing the necessary tools and notation in Chapter 2, a convergence result for a fully implicit, stress-velocity-pressure formulation of the isothermal system is established in Chapter 3; the contents of this chapter have been published as:

P.E. Farrell, **P.A. Gazca-Orozco**, E. Süli. Numerical analysis of unsteady implicitly constituted incompressible fluids: 3-field formulation. *SIAM J. Numer. Anal.*, 58(1):757–787, 2020.

The ideas of Benzi and Olshanskii [BO06] and Farrell et al [FMW19] are applied in Chapter 4, with the aim of developing a preconditioner for the 3-field formulation of the steady system, employing a discretisation based on the Scott–Vogelius pair for the velocity and pressure, which has the advantage of preserving the divergence constraint exactly (to machine precision and solver tolerances). This builds on previous work for the Navier–Stokes system [FMSW20a]; the main challenge is the development of an appropriate inner solver for the augmented stress-velocity block that is required in the implicitly constituted non-Newtonian case. The inner system presents a saddle point structure of its own, which we tackle with suitable monolithic multigrid techniques. The work presented in this chapter has been accepted for publication as:

P.E. Farrell, **P.A. Gazca-Orozco**. An augmented Lagrangian preconditioner for implicitly-constituted non-Newtonian incompressible flow. *SIAM J. Sci. Comput.* To appear.

Although the literature dealing with the numerical computation of heat-conducting fluids, including occasionally nonlinear rheology, is vast (as a very incomplete list of references we can mention [BL90, BMPT95, ZVF06, TCP10, ÇK11, LBBA13, HK15, DA16, DFG18]), relatively few consider temperature-dependent material parameters (see e.g. [HL88, EL99, FG02, VWA05, TT05, PTBC08, CSD15, OZ17, AHMS18, ABN18, AG20, AOS20]), and to our knowledge there are no rigorous convergence results available for unsteady non-Newtonian flows. The first half of Chapter 5 is devoted to filling this gap by performing a convergence analysis of a finite element discretisation of the unsteady system of forced convection introduced by Maringová and Žabenský [MZ18], thus establishing the first convergence result for the numerical approximation of unsteady heat-conducting implicitly constituted fluids. This result employs a quasi-compressible approximation and is of a highly technical nature, due to the presence of the viscous dissipation term $\mathbf{S}:\mathbf{D}(\boldsymbol{u})$. In order to present a simplified version of the argument, we then proceed to analyse the steady Oberbeck–Boussinesq system (1.24), supplemented with an implicit constitutive relation, and obtain convergence of the numerical approximations to a weak solution of the system. This last result in particular improves that of [DKS13] by employing reconstruction operators, therefore obtaining convergence in the whole admissible range $r > \frac{2d}{d+2}$, regardless of whether the elements are exactly divergence-free or not.

An augmented Lagrangian-based preconditioner (AL) for buoyancy-driven flow was already presented in [KADH18] for a stabilised $\mathbb{P}_1$–$\mathbb{P}_1$ velocity-pressure pair, in which the augmented velocity block was substituted by $A + \gamma B^\top \mathrm{diag}(M_p)^{-1} B$ and handled by GMRES preconditioned with algebraic multigrid; in that work it was shown that the AL preconditioner performed better than non-augmented variants, at least for Prandtl and Rayleigh numbers in the ranges $0.04 \leq \mathrm{Pr} \leq 1$, $500 \leq \mathrm{Ra} \leq$

10000. In the final part of Chapter 5, we present an alternative preconditioner based on the results of Chapter 4. Numerical experiments with the preconditioner will show good performance with the Navier–Stokes and power-law models for a wider range of non-dimensional numbers, even with temperature-dependent viscosity, heat conductivity, and viscous dissipation. It is remarkable that the robustness properties of the preconditioner hold in this case, given that the available parameter-robust multigrid theory pioneered by Schöberl does not apply, since the block A is non-symmetric and non-coercive. The results regarding the steady Oberbeck–Boussinesq system from Chapter 5, both the convergence result and the augmented Lagrangian preconditioner, have been submitted for publication:

P.E. Farrell, **P.A. Gazca-Orozco**, E. Süli. Finite element approximation and augmented Lagrangian preconditioning for anisothermal implicitly-constituted non-Newtonian flow. *Math. Comp.* Submitted.

Chapter 2

Preliminaries

2.1 Function spaces

Throughout this work we will assume that $\Omega \subset \mathbb{R}^d$, where $d \in \{2,3\}$ is the spatial dimension, is a bounded Lipschitz polygonal domain, and we will use standard notation for Lebesgue, Sobolev and Bochner–Sobolev spaces (for instance $(W^{k,r}(\Omega), \|\cdot\|_{W^{k,r}(\Omega)})$ and $(L^q(0,T;W^{n,r}(\Omega)), \|\cdot\|_{L^q(0,T;W^{n,r}(\Omega))})$). We will define $W_0^{k,r}(\Omega)$ for $r \in [1,\infty)$ as the closure of the space of smooth functions with compact support $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,r}(\Omega)}$ and we will denote the dual space of $W_0^{1,r}(\Omega)$ by $W^{-1,r'}(\Omega)$. Here r' is used to denote the Hölder conjugate of r, i.e. the number defined by the relation $1/r + 1/r' = 1$. The duality pairing will be written in the usual way using brackets $\langle \cdot, \cdot \rangle$. The space of traces on the boundary of functions in $W^{1,r}(\Omega)$ will be denoted by $W^{1/r',r}(\partial\Omega)$. When $r = 2$ we will write $H^1(\Omega) := W^{1,2}(\Omega)$ and $H^{-1}(\Omega) := (H_0^1(\Omega))^*$.

If X is a Banach space, $C_w([0,T];X)$ will be used to denote the space of continuous functions in time with respect to the weak topology of X. For $r \in [1,\infty)$ and $\Gamma \subset \partial\Omega$ we also define the following useful subspaces:

$$
\begin{aligned}
L_0^r(\Omega) &:= \left\{ q \in L^r(\Omega) \,:\, \int_\Omega q = 0 \right\}, \\
L^2_{\mathrm{div}}(\Omega)^d &:= \overline{\{\boldsymbol{v} \in C_0^\infty(\Omega)^d \,:\, \operatorname{div} \boldsymbol{v} = 0\}}^{\|\cdot\|_{L^2(\Omega)}}, \\
W^{1,r}_{0,\mathrm{div}}(\Omega)^d &:= \overline{\{\boldsymbol{v} \in C_0^\infty(\Omega)^d \,:\, \operatorname{div} \boldsymbol{v} = 0\}}^{\|\cdot\|_{W^{1,r}(\Omega)}}, \\
W^{1,r}_{\Gamma}(\Omega) &:= \overline{\{w \in C^\infty(\Omega)^d \,:\, w|_\Gamma = 0\}}^{\|\cdot\|_{W^{1,r}(\Omega)}}, \\
L^r_{\mathrm{sym}}(\Omega)^{d\times d} &:= \{\boldsymbol{\tau} \in L^r(\Omega)^{d\times d} \,:\, \boldsymbol{\tau}^\top = \boldsymbol{\tau}\}, \\
L^r_{\mathrm{sym,tr}}(\Omega)^{d\times d} &:= \{\boldsymbol{\tau} \in L^r_{\mathrm{sym}}(\Omega)^{d\times d} \,:\, \operatorname{tr}(\boldsymbol{\tau}) = 0\}, \\
W^{1/r',r}_{00}(\Gamma) &:= \{w \in W^{1,r}(\Omega) \,:\, w = 0 \text{ on } \partial\Omega \setminus \overline{\Gamma}\}.
\end{aligned}
$$

In the definition of the space $L^r_{\mathrm{tr}}(Q)^{d\times d}$ above, $\mathrm{tr}(\boldsymbol{\tau})$ denotes the usual matrix trace of the $d\times d$ matrix function $\boldsymbol{\tau}$. In the various estimates the letter c will denote a generic positive constant whose exact value could change from line to line, whenever the explicit dependence on the parameters is not important.

When working with boundary conditions for the velocity that allow some slip, we will employ function spaces for which only the normal component is required to be zero:

$$\begin{aligned} C^\infty_{\boldsymbol{n}}(\Omega)^d &:= \{\boldsymbol{v} \in C^\infty(\Omega)^d \,:\, \boldsymbol{v}\cdot\boldsymbol{n} = 0\},\\ C^\infty_{\boldsymbol{n},\mathrm{div}}(\Omega)^d &:= \{\boldsymbol{v} \in C^\infty_{\boldsymbol{n}}(\Omega)^d \,:\, \mathrm{div}\,\boldsymbol{v} = 0\},\\ W^{k,p}_{\boldsymbol{n}}(\Omega)^d &:= \overline{C^\infty_{\boldsymbol{n}}(\Omega)^d}^{\|\cdot\|_{W^{k,p}(\Omega)}},\\ W^{k,p}_{\boldsymbol{n},\mathrm{div}}(\Omega)^d &:= \{\boldsymbol{v} \in W^{k,p}_{\boldsymbol{n}}(\Omega)^d \,:\, \mathrm{div}\,\boldsymbol{v} = 0\},\\ W^{-1,p'}_{\boldsymbol{n}}(\Omega)^d &:= (W^{1,p}_{\boldsymbol{n}}(\Omega)^d)^*, \end{aligned} \tag{2.1}$$

where $k\in\mathbb{N}$ and $p>1$. For these spaces appropriate Poincaré and Korn inequalities also hold [BMR20] for $r>1$:

$$\|\boldsymbol{v}\|_{W^{1,r}(\Omega)^d} \leq c\|\nabla\boldsymbol{v}\|_{L^r(\Omega)} \qquad \forall\, \boldsymbol{v}\in W^{1,r}_{\boldsymbol{n}}(\Omega)^d, \tag{2.2a}$$

$$\|\nabla\boldsymbol{v}\|_{L^r(\Omega)} \leq c(\|\mathbf{D}(\boldsymbol{v})\|_{L^r(\Omega)} + \|\boldsymbol{v}\|_{L^2(\partial\Omega)}) \qquad \forall\, \boldsymbol{v}\in W^{1,r}_{\boldsymbol{n}}(\Omega)^d \text{ with } \boldsymbol{v}_\tau\in L^2(\partial\Omega)^d, \tag{2.2b}$$

The advantage of the spaces (2.1) is that they admit a Helmholtz decomposition with suitable regularity. In order to make this statement precise, suppose the following Neumann problem

$$\begin{aligned} \Delta h &= z && \text{in } \Omega,\\ \nabla h\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega,\\ \int_\Omega h &= 0, \end{aligned} \tag{2.3}$$

is $W^{2,q}$-regular; i.e. the mapping $z\in L^q_0(\Omega)\mapsto h\in W^{2,q}(\Omega)$ is well defined and bounded. Then by choosing $z=\mathrm{div}\,\boldsymbol{u}$ and denoting the corresponding solution of (2.3) by $h_{\boldsymbol{u}}$, with $\boldsymbol{u}\in W^{1,q}_{\boldsymbol{n}}(\Omega)^d$, and defining $\boldsymbol{u}_{\mathrm{div}} := \boldsymbol{u} - \nabla h_{\boldsymbol{u}}$, we have the decomposition:

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{div}} + \nabla h_{\boldsymbol{u}}, \tag{2.4}$$

where by construction $\mathrm{div}\,\boldsymbol{u}_{\mathrm{div}} = 0$. Moreover, the following estimates hold

$$\|h_{\boldsymbol{u}}\|_{W^{2,q}(\Omega)} \leq c\|\boldsymbol{u}\|_{W^{1,q}(\Omega)}, \qquad \|\boldsymbol{u}_{\mathrm{div}}\|_{W^{1,q}(\Omega)} \leq c\|\boldsymbol{u}\|_{W^{1,q}(\Omega)}, \tag{2.5a}$$

$$\|h_{\boldsymbol{u}}\|_{W^{1,q}(\Omega)} \leq c\|\boldsymbol{u}\|_{L^q(\Omega)}, \qquad \|\boldsymbol{u}_{\mathrm{div}}\|_{L^q(\Omega)} \leq c\|\boldsymbol{u}\|_{L^q(\Omega)}. \tag{2.5b}$$

The required $W^{2,q}$-regularity of the problem (2.3) is satisfied for any $q > 1$ if the domain Ω is of class $C^{1,1}$ or if the boundary is defined by piecewise smooth faces and edges given by non-intersecting closed smooth curves, with some restriction on the angles (see e.g. [Gri85, Prop. 2.5.2.3] or [MR10, Cor. 8.3.3]). However, since we wish to work with polygonal/polyhedral domains, these results are of limited use here. In two dimensions it is known that the desired regularity result holds in any convex polygonal domain [Dau92], and the following lemma states the conditions required for the result to hold in a convex three-dimensional polyhedral domain.

Lemma 2.1.1 ([MR10, Thm. 8.3.10])**.** *Suppose that $\Omega \subset \mathbb{R}^3$ is a convex polyhedral domain with edges $M_1, \dots, M_l$ and denote the interior angle between the two faces of Ω with common edge M_i by θ_i, for $i \in \{1, \dots, l\}$. Consider the problem*

$$\Delta u = f \quad \text{in } \Omega, \qquad \nabla u \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega \setminus \mathcal{S}, \tag{2.6}$$

where $\mathcal{S}$ is the set of edge points and vertices. Then the following assertions hold about the weak solution $u \in W^{1,2}(\Omega)$ of (2.6):

1. *If $f \in (W^{1,p'}(\Omega))^*$ with $p \in [2, \infty)$, then $u \in W^{1,p}(\Omega)$.*
2. *If $f \in L^p(\Omega) \cap (W^{1,2}(\Omega))^*$ with $p \in (1, 3]$ and we have that $\frac{\pi}{\theta_i} > 2 - \frac{2}{p}$ for all $i \in \{1, \dots, l\}$, then $u \in W^{2,p}(\Omega)$.*

2.2 Interpolation inequalities

The following embeddings will be useful when deriving various estimates. Assume that the Banach spaces (W_1, W_2, W_3) form an interpolation triple in the sense that, for all $v \in W_1$,

$$\|v\|_{W_2} \le c\|v\|_{W_1}^{\lambda}\|v\|_{W_3}^{1-\lambda}, \quad \text{for some } \lambda \in (0, 1),$$

and $W_1 \hookrightarrow W_2 \hookrightarrow W_3$. Then (cf. [Rou13]) $L^r(0, T; W_1) \cap L^\infty(0, T; W_3) \hookrightarrow L^{r/\lambda}(0, T; W_2)$, for $r \in [1, \infty)$ and, for all $v \in L^r(0, T; W_1) \cap L^\infty(0, T; W_3)$,

$$\|v\|_{L^{r/\lambda}(0,T;W_2)} \le c\|v\|_{L^\infty(0,T;W_3)}^{1-\lambda}\|v\|_{L^r(0,T;W_1)}^{\lambda}. \tag{2.7}$$

An example of an interpolation triple that can be combined with this result is given by the Gagliardo–Nirenberg inequality, which states that for given $p, r \in [1, \infty)$, there is a constant $c_{p,r} > 0$ such that [DiB93]:

$$\|v\|_{L^s(\Omega)} \le c_{p,r}\|\nabla v\|_{L^r(\Omega)}^{\lambda}\|v\|_{L^p(\Omega)}^{1-\lambda} \qquad \forall\, v \in W_0^{1,r}(\Omega) \cap L^p(\Omega), \tag{2.8}$$

provided that $s \in [1, \infty)$ and $\lambda \in (0, 1)$ satisfy

$$\lambda = \frac{\frac{1}{p} - \frac{1}{s}}{\frac{1}{d} - \frac{1}{r} + \frac{1}{p}}.$$

A particularly useful example can be obtained if we assume that $r > \frac{2d}{d+2}$ and take $p = 2$ and $\lambda = \frac{d}{d+2}$:

$$\|v\|_{L^{\frac{r(d+2)}{d}}(Q)} \leq c\|\nabla v\|^{\lambda}_{L^r(Q)} \|v\|^{1-\lambda}_{L^\infty(0,T;L^2(\Omega))} \quad \forall\, v \in L^r(0, T; W^{1,r}_0(\Omega)) \cap L^\infty(0, T; L^2(\Omega)). \tag{2.9}$$

2.3 Compactness and continuity in time

In this work we will use Simon's compactness lemma (see [Sim87]) instead of the usual Aubin–Lions lemma to extract convergent subsequences when taking the discretisation limit in the time-dependent problem. The reason behind this choice is that uniform estimates for the time derivative are needed in order to apply the Aubin–Lions lemma, for which the stability of the L^2-projection in Sobolev norms is essential, which imposes some restrictions on the mesh (see e.g. [CT87]).

Assume that X and H are Banach spaces such that the compact embedding $X \hookrightarrow\hookrightarrow H$ holds. Simon's lemma states that if $\mathcal{U} \subset L^p(0, T; H)$, for some $p \in [1, \infty)$, and it satisfies:

- $\mathcal{U}$ is bounded in $L^1_{\text{loc}}(0, T; X)$;
- $\int_0^{T-\epsilon} \|v(t + \epsilon, \cdot) - v(t, \cdot)\|^p_H \to 0$, as $\epsilon \to 0$, uniformly for $v \in \mathcal{U}$;

then $\mathcal{U}$ is relatively compact in $L^p(0, T; H)$.

Let X and V be reflexive Banach spaces such that $X \hookrightarrow V$ densely and let V^* be the dual space of V. The following continuity properties (see [Rou13]) will be important when identifying the initial condition:

$$v \in L^1(0, T; V^*),\ \partial_t v \in L^1(0, T; V^*) \Longrightarrow v \in C([0, T]; V^*), \tag{2.10}$$

$$v \in L^\infty(0, T; X) \cap C_w([0, T]; V) \Longrightarrow v \in C_w([0, T]; X). \tag{2.11}$$

2.4 Implicit constitutive relation and its approximation

In the mathematical analysis of these systems it is more convenient to work not with the function $\mathbf{G}$, but with its graph $\mathcal{A}$, which is introduced in the usual way:

$$(\mathbf{D}, \mathbf{S}) \in \mathcal{A}(\cdot) \quad \Longleftrightarrow \quad \mathbf{G}(\cdot, \mathbf{S}, \mathbf{D}) = \mathbf{0}. \tag{2.12}$$

The essential assumption will be that $\mathcal{A}$ is a *maximal monotone r-graph* for some $r > 1$, which means that the following properties hold for almost every $z \in Q$:

(A1) [*$\mathcal{A}$ includes the origin*] $(\mathbf{0}, \mathbf{0}) \in \mathcal{A}(z)$.

(A2) [*$\mathcal{A}$ is a monotone graph*] For every $(\mathbf{D}_1, \mathbf{S}_1), (\mathbf{D}_2, \mathbf{S}_2) \in \mathcal{A}(z)$,

$$(\mathbf{S}_1 - \mathbf{S}_2) : (\mathbf{D}_1 - \mathbf{D}_2) \geq 0.$$

(A3) [*$\mathcal{A}$ is maximal monotone*] If $(\mathbf{D}, \mathbf{S}) \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}}$ is such that

$$(\hat{\mathbf{S}} - \mathbf{S}) : (\hat{\mathbf{D}} - \mathbf{D}) \geq 0 \quad \text{for all } (\hat{\mathbf{D}}, \hat{\mathbf{S}}) \in \mathcal{A}(z),$$

then $(\mathbf{D}, \mathbf{S}) \in \mathcal{A}(z)$.

(A4) [*$\mathcal{A}$ is an r-graph*] There is a non-negative function $m \in L^1(Q)$ and a constant $c > 0$ such that

$$\mathbf{S} : \mathbf{D} \geq -m + c(|\mathbf{D}|^r + |\mathbf{S}|^{r'}) \quad \text{for all } (\mathbf{D}, \mathbf{S}) \in \mathcal{A}(z).$$

(A5) [*Measurability*] The set-valued map $z \mapsto \mathcal{A}(z)$ is $\mathcal{L}(Q)$–$(\mathcal{B}(\mathbb{R}^{d\times d}_{\text{sym}}) \otimes \mathcal{B}(\mathbb{R}^{d\times d}_{\text{sym}}))$ measurable; here $\mathcal{L}(Q)$ denotes the family of Lebesgue measurable subsets of Q and $\mathcal{B}(\mathbb{R}^{d\times d}_{\text{sym}})$ is the family of Borel subsets of $\mathbb{R}^{d\times d}_{\text{sym}}$.

(A6) [*Compatibility*] For any $(\mathbf{D}, \mathbf{S}) \in \mathcal{A}(z)$ we have that

$$\operatorname{tr}(\mathbf{D}) = 0 \Longleftrightarrow \operatorname{tr}(\mathbf{S}) = 0.$$

Assumption (A6) was not included in the original works [BGMŚ09, BGMŚ12, DKS13], but it is needed for consistency with the physical property that $\mathbf{S}$ is traceless if and only if the velocity field is divergence-free (see the discussion in [Tsc18]). To illustrate this point, consider an implicit constitutive relation describing a Bingham fluid:

$$\mathbf{G}(\mathbf{S}, \mathbf{D}) = (\tau_* + (|\mathbf{S}| - \tau_*)^+)\mathbf{D} - (|\mathbf{S}| - \tau_*)^+\mathbf{S} = \mathbf{0}, \tag{2.13}$$

where $\tau_* > 0$ is the yield stress. The relation (2.13) induces a perfectly well defined graph in $\mathbb{R}^{d\times d}_{\text{sym,tr}} \times \mathbb{R}^{d\times d}_{\text{sym,tr}}$, but when considered in the bigger space $\mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}}$ an inconsistency appears: we could have admissible physical states with $\operatorname{div} \boldsymbol{u} = 0$ but $\operatorname{tr} \mathbf{S} \neq 0$ (e.g. by taking $|\mathbf{S}| \leq \tau_*$ this does not lead to a contradiction), which does not seem desirable from the modelling point of view. This could be addressed by extending the relation to $\mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}}$ in the following way:

$$\tilde{\mathbf{G}}(\mathbf{S}, \mathbf{D}) = (\tau_* + (|\mathbf{S}_\delta| - \tau_*)^+)(\mathbf{D} - \tfrac{1}{d}\operatorname{tr}(\mathbf{S})\mathbf{I}) - (|\mathbf{S}_\delta| - \tau_*)^+\mathbf{S}_\delta = 0, \tag{2.14}$$

where $\mathbf{S}_\delta := \mathbf{S} - \frac{1}{d}\operatorname{tr}(\mathbf{S})\mathbf{I}$. The graph $\tilde{\mathcal{A}}$ induced by (2.14) coincides with the graph $\mathcal{A}$ induced by (2.13) when the matrices are traceless, it does not present the inconsistency mentioned above, and it can be proven to satisfy the assumptions (A1)–(A5) (see [Tsc18]). However, this expression is more involved than the usual formulations of such fluids. This problem is fixed by either introducing the assumption (A6) or by working instead with graphs defined on $\mathbb{R}^{d\times d}_{\mathrm{sym,tr}} \times \mathbb{R}^{d\times d}_{\mathrm{sym,tr}}$. The latter could be used when performing the PDE analysis and the proofs would remain intact, but it is not so convenient when dealing with numerical approximations, given that one cannot always guarantee that the discrete velocities $\boldsymbol{u}^n$ are pointwise divergence-free (and so $\mathbf{D}(\boldsymbol{u}^n)$ does not necessarily belong to $\mathbb{R}^{d\times d}_{\mathrm{sym,tr}}$). We therefore choose to introduce this extra assumption.

A very important consequence of Assumption (A5) (see [Tsc18]) is the existence of a measurable function (usually called a *selection*) $\boldsymbol{\mathcal{D}}\colon Q\times\mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that $(\boldsymbol{\mathcal{D}}(z,\boldsymbol{\sigma}),\boldsymbol{\sigma}) \in \mathcal{A}(z)$ for all $\boldsymbol{\sigma}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$. In the existence results it will be useful to approximate the selection using smooth functions. To that end, let us define the mollification:

$$\boldsymbol{\mathcal{D}}^k(\cdot,\boldsymbol{\sigma}) := \int_{\mathbb{R}^{d\times d}_{\mathrm{sym}}} \boldsymbol{\mathcal{D}}(\cdot,\boldsymbol{\sigma}-\boldsymbol{\tau})\rho^k(\boldsymbol{\tau})\,\mathrm{d}\boldsymbol{\tau}, \tag{2.15}$$

where $\rho^k(\boldsymbol{\tau}) = k^{d^2}\rho(k\boldsymbol{\tau})$, $k\in\mathbb{N}$, and $\rho\in C^\infty_0(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ is a mollification kernel. It is possible to check (see e.g. [Tsc18]) that this mollification satisfies analogous monotonicity and coercivity properties to those of the selection $\boldsymbol{\mathcal{D}}$, i.e. we have that

- For every $\boldsymbol{\tau}_1,\boldsymbol{\tau}_2\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$ and for almost every $z\in Q$ the monotonicity condition

$$(\boldsymbol{\mathcal{D}}^k(z,\boldsymbol{\tau}_1) - \boldsymbol{\mathcal{D}}^k(z,\boldsymbol{\tau}_2)) : (\boldsymbol{\tau}_1-\boldsymbol{\tau}_2) \geq 0 \tag{2.16}$$

  holds.

- There is a constant $c_* > 0$ and a non-negative function $g\in L^1(Q)$ such that for all $k\in\mathbb{N}$, for every $\boldsymbol{\tau}\in\mathbb{R}^{d\times d}$, and for almost every $z\in Q$ we have

$$\boldsymbol{\tau} : \boldsymbol{\mathcal{D}}^k(z,\boldsymbol{\tau}) \geq -g(z) + c_*(|\boldsymbol{\tau}|^{r'} + |\boldsymbol{\mathcal{D}}^k(z,\boldsymbol{\tau})|^r). \tag{2.17}$$

- For any sequence $\{\mathbf{S}_k\}_{k\in\mathbb{N}}$ bounded in $L^{r'}(Q)^{d\times d}$, we have for arbitrary $\mathbf{B}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\phi\in C^\infty_0(Q)$ with $\phi\geq 0$:

$$\liminf_{k\to\infty}\int_Q (\boldsymbol{\mathcal{D}}^k(\cdot,\mathbf{S}^k) - \boldsymbol{\mathcal{D}}(\cdot,\mathbf{B})) : (\mathbf{S}^k - \mathbf{B})\phi(\cdot) \geq 0. \tag{2.18}$$

It is important to remark that (2.16), (2.17) and (2.18) are the essential properties; the explicit form (2.15) of the approximation to the selection is not very important. There are other ways to achieve the same result; for instance piecewise affine interpolation or a generalised Yosida approximation could also be used (see [ST19, Tsc18]). The following is a localized version of Minty's lemma that will aid in the identification of the implicit constitutive relation (for a proof see [BGM+12]).

Lemma 2.4.1. *Let $\mathcal{A}$ be a maximal monotone r-graph satisfying (A1)–(A4) for some $r > 1$. Suppose that $\{\mathbf{D}^n\}_{n\in\mathbb{N}}$ and $\{\mathbf{S}^n\}_{n\in\mathbb{N}}$ are sequences of functions defined on a measurable set $\hat{Q} \subset Q$, such that:*

$$\begin{aligned}
(\mathbf{D}^n, \mathbf{S}^n) &\in \mathcal{A}(\cdot) && \text{a.e. in } \hat{Q}, \\
\mathbf{D}^n &\rightharpoonup \mathbf{D} && \text{weakly in } L^r(\hat{Q})^{d\times d}, \\
\mathbf{S}^n &\rightharpoonup \mathbf{S} && \text{weakly in } L^{r'}(\hat{Q})^{d\times d}, \\
\limsup_{n\to\infty} \int_{\hat{Q}} \mathbf{S}^n : \mathbf{D}^n &\leq \int_{\hat{Q}} \mathbf{S} : \mathbf{D}.
\end{aligned}$$

Then,

$$(\mathbf{D}, \mathbf{S}) \in \mathcal{A}(\cdot) \qquad \text{a.e. in } \hat{Q}.$$

The requirement that $(\mathbf{D}^n, \mathbf{S}^n) \in \mathcal{A}(\cdot)$ in Lemma 2.4.1 can be relaxed when using certain kinds of graph approximations. For instance, when using the generalised Yosida approximation then one only requires that the approximate solutions $(\mathbf{D}^n, \mathbf{S}^n)$ belong to the approximate graph $\mathcal{A}^n$, which can simplify some convergence proofs. This approximation is defined through (c.f. [Tsc18])

$$\boldsymbol{\mathcal{D}}^n(x, \mathbf{S}) := \{\mathbf{D} \in \mathbb{R}^{d\times d}_{\text{sym}} : (\mathbf{D}, \mathbf{S}) \in \mathcal{A}^n(x)\}, \tag{2.19}$$

where the approximate graph $\mathcal{A}^n$ is defined as follows

$$\mathcal{A}^n(x) := \{\big(\mathbf{D}, \mathbf{S} + \tfrac{1}{n}|\mathbf{D}|^{r-2}\mathbf{D}\big) \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}} : (\mathbf{D}, \mathbf{S}) \in \mathcal{A}(x)\}, \tag{2.20}$$

where $x \in \Omega$. The relation (2.19) defines in fact a single-valued function that can be employed in the definition of a finite element formulation.

Lemma 2.4.2. *Let $\mathcal{A}$ be a maximal monotone r-graph satisfying (A1)–(A4) for some $r > 1$, and let $\mathcal{A}^n$ be the generalised Yosida approximation defined by (2.20). Suppose that $\{\mathbf{D}^n\}_{n\in\mathbb{N}}$ and $\{\mathbf{S}^n\}_{n\in\mathbb{N}}$ are sequences of functions defined on a measurable set $\hat{Q} \subset Q$, such that:*

$$(\mathbf{D}^n, \mathbf{S}^n) \in \mathcal{A}^n(\cdot) \qquad \text{a.e. in } \hat{Q},$$

$$
\begin{aligned}
\mathbf{D}^n &\rightharpoonup \mathbf{D} && \textit{weakly in } L^r(\hat{Q})^{d\times d},\\
\mathbf{S}^n &\rightharpoonup \mathbf{S} && \textit{weakly in } L^{r'}(\hat{Q})^{d\times d},\\
\limsup_{n\to\infty} \int_{\hat{Q}} \mathbf{S}^n : \mathbf{D}^n &\le \int_{\hat{Q}} \mathbf{S} : \mathbf{D}.
\end{aligned}
$$

Then,

$$
(\mathbf{D}, \mathbf{S}) \in \mathcal{A}(\cdot) \qquad \textit{a.e. in } \hat{Q}.
$$

2.5 Finite element approximation

In this section, the notation and assumptions regarding the finite element approximation will be presented. Consider a family of triangulations $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ of Ω satisfying the following assumptions:

- (Affine equivalence). Given $n \in \mathbb{N}$ and an element $K \in \mathcal{T}_n$, there is an affine invertible mapping $\boldsymbol{F}_K \colon K \to \hat{K}$, where $\hat{K}$ is the closed standard reference simplex in $\mathbb{R}^d$.
- (Shape-regularity). There is a constant c_τ, independent of n, such that

$$
h_K \le c_\tau \rho_K \qquad \text{for every } K \in \mathcal{T}_n,\ n \in \mathbb{N},
$$

  where $h_K := \operatorname{diam}(K)$ and ρ_K is the diameter of the largest inscribed ball.
- The mesh size $h_n := \max_{K\in\mathcal{T}_n} h_K$ tends to zero as $n \to \infty$.

Let V be defined as $W^{1,\infty}_0(\Omega)^d$ or $W^{1,\infty}_{\boldsymbol{n}}(\Omega)^d$ depending on whether the boundary condition $\boldsymbol{u} = \boldsymbol{0}$ or $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ is imposed on $\partial\Omega$. For problems involving temperature the aim is to consider Dirichlet boundary conditions imposed on a (relatively open) subset Γ_D of $\partial\Omega$. We thus define the conforming finite element spaces associated with the triangulation $\mathcal{T}_n$ as follows:

$$
\begin{aligned}
V^n &:= \left\{ \boldsymbol{v} \in V \,:\, \boldsymbol{v}|_K \circ \boldsymbol{F}_K^{-1} \in \hat{\mathbb{P}}_{\mathbb{V}},\ K \in \mathcal{T}_n \right\},\\
M^n &:= \left\{ q \in L^\infty(\Omega) \,:\, q|_K \circ \boldsymbol{F}_K^{-1} \in \hat{\mathbb{P}}_{\mathbb{M}},\ K \in \mathcal{T}_n \right\},\\
\Sigma^n &:= \left\{ \boldsymbol{\sigma} \in L^\infty_{\mathrm{sym}}(\Omega)^{d\times d} \,:\, \boldsymbol{\sigma}|_K \circ \boldsymbol{F}_K^{-1} \in \hat{\mathbb{P}}_{\mathbb{S}},\ K \in \mathcal{T}_n \right\},\\
U^n &:= \left\{ w \in W^{1,\infty}_{\Gamma_D}(\Omega) \,:\, w|_K \circ \boldsymbol{F}_K^{-1} \in \hat{\mathbb{P}}_{\mathbb{U}},\ K \in \mathcal{T}_n \right\},
\end{aligned}
$$

where $\hat{\mathbb{P}}_{\mathbb{V}} \subset W^{1,\infty}(\hat{K})^d$, $\hat{\mathbb{P}}_{\mathbb{M}} \subset L^\infty(\hat{K})$, $\hat{\mathbb{P}}_{\mathbb{S}} \subset L^\infty_{\mathrm{sym}}(\hat{K})^{d\times d}$, and $\hat{\mathbb{P}}_{\mathbb{U}} \subset W^{1,\infty}(\hat{K})$ are finite-dimensional polynomial subspaces on the reference simplex $\hat{K}$. Each of these spaces will be assumed to have a finite and locally supported basis. As in the continuous case, it will be useful to introduce the following finite-dimensional subspaces for $r > 1$:

$$M_0^n := M^n \cap L_0^{r'}(\Omega),$$
$$\Sigma_{\mathrm{tr}}^n := \Sigma^n \cap L_{\mathrm{sym,tr}}^r(\Omega)^{d\times d},$$
$$V_{\mathrm{div}}^n := \left\{ \boldsymbol{v} \in V^n \; : \; \int_\Omega q \,\mathrm{div}\, \boldsymbol{v} = 0, \quad \forall\, q \in M^n \right\}.$$

Assumption 2.5.1 (Approximability). *For every $s \in [1,\infty)$ we have that*

$$\inf_{\overline{\boldsymbol{v}} \in V^n} \|\boldsymbol{v} - \overline{\boldsymbol{v}}\|_{W^{1,s}(\Omega)} \to 0 \quad \text{as } n \to \infty \quad \forall\, \boldsymbol{v} \in W_0^{1,s}(\Omega)^d,$$
$$\inf_{\overline{q} \in M^n} \|q - \overline{q}\|_{L^s(\Omega)} \to 0 \quad \text{as } n \to \infty \quad \forall\, q \in L^s(\Omega),$$
$$\inf_{\overline{\boldsymbol{\sigma}} \in \Sigma^n} \|\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}\|_{L^s(\Omega)} \to 0 \quad \text{as } n \to \infty \quad \forall\, \boldsymbol{\sigma} \in L^s(\Omega)^{d\times d},$$
$$\inf_{\overline{w} \in U^n} \|w - \overline{w}\|_{W^{1,s}(\Omega)} \to 0 \quad \text{as } n \to \infty \quad \forall\, w \in W_{\Gamma_D}^{1,s}(\Omega).$$

When employing boundary conditions for the velocity that allow slip, the first condition holds instead for all $\boldsymbol{v} \in W_{\boldsymbol{n}}^{1,s}(\Omega)^d$.

Assumption 2.5.2 (Fortin Projector Π_Σ^n). *For each $n \in \mathbb{N}$ there is a linear projector $\Pi_\Sigma^n \colon L_{\mathrm{sym}}^1(\Omega)^{d\times d} \to \Sigma^n$ such that:*

- *(Preservation of divergence). For every $\boldsymbol{\sigma} \in L^1(\Omega)^{d\times d}$ we have*

$$\int_\Omega \boldsymbol{\sigma} : \mathbf{D}(\boldsymbol{v}) = \int_\Omega \Pi_\Sigma^n(\boldsymbol{\sigma}) : \mathbf{D}(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V_{\mathrm{div}}^n.$$

- *(L^s–stability). For every $s \in (1,\infty)$ there is a constant $c > 0$, independent of n, such that:*

$$\|\Pi_\Sigma^n \boldsymbol{\sigma}\|_{L^s(\Omega)} \leq c \|\boldsymbol{\sigma}\|_{L^s(\Omega)} \qquad \forall\, \boldsymbol{\sigma} \in L_{\mathrm{sym}}^s(\Omega)^{d\times d}.$$

Assumption 2.5.3 (Fortin Projector Π_V^n). *For each $n \in \mathbb{N}$ there is a linear projector $\Pi_V^n \colon W_0^{1,1}(\Omega)^d \to V^n$ such that the following properties hold:*

- *(Preservation of divergence). For every $\boldsymbol{v} \in W_0^{1,1}(\Omega)^d$ we have*

$$\int_\Omega q \,\mathrm{div}\, \boldsymbol{v} = \int_\Omega q \,\mathrm{div}(\Pi_V^n \boldsymbol{v}) \quad \forall\, q \in M^n.$$

- *($W^{1,s}$–stability). For every $s \in (1,\infty)$ there is a constant $c > 0$, independent of n, such that:*

$$\|\Pi_V^n \boldsymbol{v}\|_{W^{1,s}(\Omega)} \leq c \|\boldsymbol{v}\|_{W^{1,s}(\Omega)} \qquad \forall\, \boldsymbol{v} \in W_0^{1,s}(\Omega)^d.$$

When employing boundary conditions that allow slip, the space $W^{1,s}_{\boldsymbol{n}}(\Omega)^d$ *is used instead.*

Assumption 2.5.4 (Projector Π^n_M). *For each* $n \in \mathbb{N}$ *there is a linear projector* $\Pi^n_M \colon L^1(\Omega) \to M^n$ *such that for all* $s \in (1,\infty)$ *there is a constant* $c > 0$*, independent of* n*, such that:*

$$\|\Pi^n_M q\|_{L^s(\Omega)} \leq c\|q\|_{L^s(\Omega)} \qquad \forall\, q \in L^s(\Omega).$$

Assumption 2.5.5 (Projector Π^n_U). *For each* $n \in \mathbb{N}$ *there is a linear projector* $\Pi^n_U \colon W^{1,1}_{\Gamma_D}(\Omega) \to U^n$ *such that for all* $s \in (1,\infty)$ *there is a constant* $c > 0$*, independent of* n*, such that:*

$$\|\Pi^n_U w\|_{W^{1,s}(\Omega)} \leq c\|w\|_{W^{1,s}(\Omega)} \qquad \forall\, w \in W^{1,s}_{\Gamma_D}(\Omega).$$

It is not difficult to show that the approximability and stability properties imply that for $s \in [1,\infty)$ we have:

$$\begin{aligned}
\|\boldsymbol{\sigma} - \Pi^n_\Sigma \boldsymbol{\sigma}\|_{L^s(\Omega)} &\to 0 \quad \text{as } n\to\infty \quad \forall\, \boldsymbol{\sigma} \in L^s_{\mathrm{sym}}(\Omega)^{d\times d},\\
\|\boldsymbol{v} - \Pi^n_V \boldsymbol{v}\|_{W^{1,s}(\Omega)} &\to 0 \quad \text{as } n\to\infty \quad \forall\, \boldsymbol{v} \in W^{1,s}_0(\Omega)^d,\\
\|q - \Pi^n_M q\|_{L^s(\Omega)} &\to 0 \quad \text{as } n\to\infty \quad \forall\, q \in L^s(\Omega),\\
\|w - \Pi^n_U w\|_{W^{1,s}(\Omega)} &\to 0 \quad \text{as } n\to\infty \quad \forall\, w \in W^{1,s}_{\Gamma_D}(\Omega).
\end{aligned} \tag{2.21}$$

Remark 2.5.6. *A very important consequence of the previous assumptions is the existence, for every* $s \in (1,\infty)$*, of two positive constants* $\beta_s, \gamma_s > 0$*, independent of* n*, such that the following discrete inf-sup conditions hold (c.f. [EG16]):*

$$\inf_{q\in M^n_0} \sup_{\boldsymbol{v}\in V^n} \frac{\int_\Omega q \,\mathrm{div}\, \boldsymbol{v}}{\|\boldsymbol{v}\|_{W^{1,s}(\Omega)}\|q\|_{L^{s'}(\Omega)}} \geq \beta_s, \tag{2.22}$$

$$\inf_{\boldsymbol{v}\in V^n_{\mathrm{div}}} \sup_{\boldsymbol{\tau}\in \Sigma^n} \frac{\int_\Omega \boldsymbol{\tau} : \mathbf{D}(\boldsymbol{v})}{\|\boldsymbol{\tau}\|_{L^{s'}(\Omega)}\|\boldsymbol{v}\|_{W^{1,s}(\Omega)}} \geq \gamma_s. \tag{2.23}$$

Example 2.5.7. *There are several pairs of velocity-pressure spaces known to satisfy the stability Assumptions 2.5.1 and 2.5.3 (see also the discussion in [Tsc18]). They include the conforming Crouzeix–Raviart element, the MINI element, the* $\mathbb{P}_2$*–*$\mathbb{P}_0$ *element and the Taylor–Hood element* $\mathbb{P}_k$*–*$\mathbb{P}_{k-1}$ *(see [BBDR12, BBF13, DKS13, GS03, CR73]). In addition to stability, the Scott–Vogelius element* $\mathbb{P}_k$*–*$\mathbb{P}^{\mathrm{disc}}_{k-1}$ *also satisfies the property that the discretely divergence-free velocities are pointwise divergence-free (the stability can be guaranteed by assuming for example that the mesh has been barycentrically refined, see [Qin94, Zha05, SV85]); another example of a velocity-pressure pair* *with this property is given by the Guzmán–Neilan element [GN14b, GN14a]. To satisfy Assumptions 2.5.4 and 2.5.5, one could use the Scott–Zhang interpolant [SZ90].*

Sometimes it is easier to prove the inf-sup condition directly. For example, if the space of discrete stresses consists of discontinuous $\mathbb{P}_{k-1}$ polynomials (with $k \geq 2$):

$$\Sigma^n = \{\boldsymbol{\sigma} \in L^\infty_{\mathrm{sym}}(\Omega)^{d\times d} \,:\, \boldsymbol{\sigma}|_K \in \mathbb{P}_{k-1}(K)^{d\times d},\ \textit{for all } K \in \mathcal{T}_n\},$$

and we have that $\mathbf{D}(V^n) \subset \Sigma^n$ (e.g. we could take the Scott–Vogelius element $\mathbb{P}_k$–$\mathbb{P}^{\mathrm{disc}}_{k-1}$ for the velocity and the pressure), then the inf-sup condition follows from the fact that for $s \in (1,\infty)$ there is a constant $c > 0$, independent of n, such that for any $\boldsymbol{\sigma} \in \Sigma^n$ there is a $\boldsymbol{\tau} \in \Sigma^n$ such that [San98]:

$$\int_\Omega \boldsymbol{\tau} : \boldsymbol{\sigma} = \|\boldsymbol{\sigma}\|^s_{L^s(\Omega)} \quad \textit{and} \quad \|\boldsymbol{\tau}\|_{L^{s'}(\Omega)} \leq c\|\boldsymbol{\sigma}\|^{s-1}_{L^s(\Omega)}.$$

In case a continuous piecewise polynomial approximation of the stress is preferred, for the two-dimensional problem one could use the conforming Crouzeix–Raviart element for the discrete velocity and pressure and the following space for the stress [Rua94]:

$$\Sigma^n = \{\boldsymbol{\sigma} \in C(\overline{\Omega})^{2\times 2} \,:\, \boldsymbol{\sigma} = \boldsymbol{\sigma}^\top,\, \boldsymbol{\sigma}|_K \in (\mathbb{P}_1(K) \oplus \mathcal{B})^{2\times 2},\ \textit{for all } K \in \mathcal{T}_n\},$$

where

$$\mathcal{B} := \textit{span}\,\{\lambda_1^2\lambda_2\lambda_3, \lambda_1\lambda_2^2\lambda_3, \lambda_1\lambda_2\lambda_3^2\},$$

and $\{\lambda_j\}_{j=1}^3$ are barycentric coordinates on K (for the three-dimensional analogue see [Rua96]).

Remark 2.5.8. *If the discretely divergence-free velocities are in fact exactly divergence free, i.e. if $\boldsymbol{v} \in V^n_{\mathrm{div}}$ implies that $\operatorname{div} \boldsymbol{v} = 0$ pointwise, and $\mathbf{D}(V^n) \subset \Sigma^n$, then the stress-velocity inf-sup condition (2.23) also holds for the subspace of traceless stresses. Consequently, fewer degrees of freedom are needed to compute the stress unknowns.*

2.6 Time discretisation

In this section we will describe the notation that will be used when performing the time discretisation of the problem. Let $\{\tau_m\}_{m\in\mathbb{N}}$ be a sequence of time steps such that $T/\tau_m \in \mathbb{N}$ and $\tau_m \to 0$, as $m \to \infty$. For each $m \in \mathbb{N}$ we define the equidistant grid:

$$\{t_j^m\}_{j=0}^{T/\tau_m}, \qquad t_j = t_j^m := j\tau_m.$$

This can be used to define the parabolic cylinders $Q_i^j := (t_i, t_j) \times \Omega$, where $0 \leq i \leq j \leq T/\tau_m$. Also, given a set of functions $\{v_j\}_{j=0}^{T/\tau_m}$ belonging to a Banach space X, we can define the piecewise constant interpolant $\overline{v} \in L^\infty(0,T;X)$ as:

$$\overline{v}(t) := v_j, \qquad t \in (t_{j-1}, t_j], \qquad j \in \{1, \dots, T/\tau_m\}, \tag{2.24}$$

and the piecewise linear interpolant $\tilde{v} \in C([0,T];X)$ as:

$$\tilde{v}(t) := \frac{t - t_{j-1}}{\tau_m} v_j + \frac{t_j - t}{\tau_m} v_{j-1}, \qquad t \in [t_{j-1}, t_j], \quad j \in \{1, \dots, T/\tau_m\}. \tag{2.25}$$

For a given function $g \in L^p(0,T;X)$, with $p \in [1, \infty)$, we define the time averages:

$$g_j(\cdot) := \frac{1}{\tau_m} \int_{t_{j-1}}^{t_j} g(t, \cdot) \, \mathrm{d}t, \quad j \in \{1, \dots, T/\tau_m\}. \tag{2.26}$$

Then the piecewise constant interpolant $\overline{g}$ defined by (2.24) satisfies [Rou13]:

$$\|\overline{g}\|_{L^p(0,T;X)} \leq \|g\|_{L^p(0,T;X)}, \tag{2.27}$$

and

$$\overline{g} \to g \text{ strongly in } L^p(0,T;X), \text{ as } m \to \infty. \tag{2.28}$$

Chapter 3

Implicitly Constituted Fluids: Isothermal Case

The goal of this chapter is to prove convergence of a three-field finite element approximation of the following system:

$$
\begin{aligned}
\partial_t \boldsymbol{u} - \operatorname{div}(\mathbf{S} - \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \boldsymbol{f} \qquad & \text{in } (0,T)\times\Omega, \\
\operatorname{div} \boldsymbol{u} = 0 \qquad & \text{in } (0,T)\times\Omega, \\
(\mathbf{D}(\boldsymbol{u}), \mathbf{S}) \in \mathcal{A}(\cdot) \qquad & \text{a.e. in } (0,T)\times\Omega, \\
\boldsymbol{u} = \boldsymbol{0} \qquad & \text{on } (0,T)\times\partial\Omega, \\
\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0(\cdot) \qquad & \text{in } \Omega,
\end{aligned}
\tag{3.1}
$$

where $\mathcal{A}(\cdot)$ satisfies (A1)–(A6). The next section introduces the notation and tools that will be useful in the analysis of the discrete problem.

3.1 Weak formulation

In this section we will present a weak formulation for the problem (3.1), where now we assume that $\boldsymbol{f} \in L^{r'}(0,T;W^{-1,r'}(\Omega)^d)$, $\boldsymbol{u}_0 \in L^2_{\operatorname{div}}(\Omega)^d$ and the graph $\mathcal{A}$ satisfies the assumptions (A1)–(A6) for some $r > \frac{2d}{d+2}$. Similarly to previous works on the analysis of implicitly constituted fluids, a Lipschitz truncation technique will be required when proving that the limit of the sequence of approximate solutions satisfies the constitutive relation. The theory of Lipschitz truncation for time-dependent problems is not as well developed as in the steady case; here it will be necessary to work locally and the equation plays a vital role (several versions of parabolic Lipschitz truncation have appeared in the literature, see e.g. [DRW10, BGMŚ12, BDS13, DSSV17]). Since the pressure will not be present in the weak formulation, it will be more convenient to use the construction developed in [BDS13] because it preserves the solenoidality of the velocity. The following lemma states the main properties of this solenoidal Lipschitz truncation.

Lemma 3.1.1. ([BDS13, ST19]) *Let $p \in (1,\infty)$, $\sigma \in (1, \min(p,p'))$ and let $Q_0 = I_0 \times B_0 \subset \mathbb{R} \times \mathbb{R}^3$ be a parabolic cylinder, where I_0 is an open interval and B_0 is an open ball. Denote by αQ_0, where $\alpha > 0$, the α-scaled version of Q_0 keeping the barycenter the same. Suppose $\{\boldsymbol{e}^l\}_{l\in\mathbb{N}}$ is a sequence of divergence-free functions that is uniformly bounded in $L^\infty(I_0; L^\sigma(B_0)^d)$ and converges to zero weakly in $L^p(I_0; W^{1,p}(B_0)^d)$ and strongly in $L^\sigma(Q_0)^d$. Let $\{\mathsf{G}_1^l\}_{l\in\mathbb{N}}$ and $\{\mathsf{G}_2^l\}_{l\in\mathbb{N}}$ be sequences that converge to zero weakly in $L^{p'}(Q_0)^{d\times d}$ and strongly in $L^\sigma(Q_0)^{d\times d}$, respectively. Define $\mathsf{G}^l := \mathsf{G}_1^l + \mathsf{G}_2^l$ and suppose that, for any $l \in \mathbb{N}$, the equation*

$$\int_{Q_0} \partial_t \boldsymbol{e}^l \cdot \boldsymbol{w} = \int_{Q_0} \mathsf{G}^l : \nabla \boldsymbol{w} \quad \forall\, \boldsymbol{w} \in C^\infty_{0,\mathrm{div}}(Q_0)^d. \tag{3.2}$$

is satisfied. Then there is a number $j_0 \in \mathbb{N}$, a sequence $\{\lambda_{l,j}\}_{l,j\in\mathbb{N}}$ with $2^{2^j} \leq \lambda_{l,j} \leq 2^{2^{j+1}-1}$, a sequence of functions $\{\boldsymbol{e}^{l,j}\}_{l,j\in\mathbb{N}} \subset L^1(Q_0)^d$, a sequence of open sets $\mathcal{B}_{\lambda_{l,j}} \subset Q_0$, for $l, j \in \mathbb{N}$, and a function $\zeta \in C_0^\infty(\frac{1}{6}Q_0)$ with $\mathbb{1}_{\frac{1}{8}Q_0} \leq \zeta \leq \mathbb{1}_{\frac{1}{6}Q_0}$ with the following properties:

1. *$\boldsymbol{e}^{l,j} \in L^q(\frac{1}{4}I_0; W^{1,q}_{0,\mathrm{div}}(\frac{1}{6}B_0)^d)$ for any $q \in [1,\infty)$ and $\operatorname{supp}(\boldsymbol{e}^{l,j}) \subset \frac{1}{6}Q_0$, for any $j \geq j_0$ and any $l \in \mathbb{N}$;*

2. *$\boldsymbol{e}^{l,j} = \boldsymbol{e}^j$ on $\frac{1}{8}Q_0 \setminus \mathcal{B}_{\lambda_{l,j}}$, for any $j \geq j_0$ and any $l \in \mathbb{N}$;*

3. *There is a constant $c > 0$ such that*

$$\limsup_{l\to\infty} \lambda_{l,j}^p |\mathcal{B}_{\lambda_{l,j}}| \leq c2^{-j}, \quad \text{for any } j \geq j_0;$$

4. *For $j \geq j_0$ fixed, we have as $l \to \infty$:*

$$\begin{aligned} \boldsymbol{e}^{l,j} &\to \boldsymbol{0}, && \text{strongly in } L^\infty(\tfrac{1}{4}Q_0)^d, \\ \nabla \boldsymbol{e}^{l,j} &\rightharpoonup \boldsymbol{0}, && \text{weakly in } L^q(\tfrac{1}{4}Q_0)^{d\times d}, \quad \forall\, q \in [1,\infty); \end{aligned}$$

5. *There is a constant $c > 0$ such that:*

$$\limsup_{l\to\infty} \left| \int_{Q_0} \mathsf{G}^l : \nabla \boldsymbol{e}^{l,j} \right| \leq c2^{-j}, \quad \text{for any } j \geq j_0;$$

6. *There is a constant $c > 0$ such that for any $\boldsymbol{H} \in L^{p'}(\frac{1}{6}Q_0)^{d\times d}$:*

$$\limsup_{l\to\infty} \left| \int_{Q_0} (\mathsf{G}_1^l + \boldsymbol{H}) : \nabla \boldsymbol{e}^{l,j} \zeta \mathbb{1}_{\mathcal{B}^c_{\lambda_{l,j}}} \right| \leq c2^{-j/p}, \quad \text{for any } j \geq j_0.$$

3.1.1 Mixed formulation and time–space discretisation

Before we present the weak formulation, let us define

$$\check{r} := \min\left\{\frac{r(d+2)}{2d}, r'\right\}.$$

The weak formulation for (3.1) then reads as follows.
Formulation $\check{\text{A}}$. Find functions

$$\begin{aligned}
\mathbf{S} &\in L^{r'}_{\mathrm{sym,tr}}(Q)^{d\times d},\\
\boldsymbol{u} &\in L^{r}(0,T;W^{1,r}_{0,\mathrm{div}}(\Omega)^d)\cap L^\infty(0,T;L^2_{\mathrm{div}}(\Omega)^d),\\
\partial_t\boldsymbol{u} &\in L^{\check{r}}(0,T;(W^{1,\check{r}'}_{0,\mathrm{div}}(\Omega)^d)^*),
\end{aligned}$$

such that

$$\begin{gathered}
\langle\partial_t\boldsymbol{u},\boldsymbol{v}\rangle+\int_\Omega(\mathbf{S}-\boldsymbol{u}\otimes\boldsymbol{u}):\mathbf{D}(\boldsymbol{v})=\langle\boldsymbol{f},\boldsymbol{v}\rangle\qquad\forall\,\boldsymbol{v}\in W^{1,\check{r}'}_{0,\mathrm{div}}(\Omega)^d,\ \text{a.e. } t\in(0,T),\\
(\mathbf{D}(\boldsymbol{u}),\mathbf{S})\in\mathcal{A}(\cdot),\ \text{a.e. in } (0,T)\times\Omega,\\
\operatorname*{ess\,lim}_{t\to0^+}\|\boldsymbol{u}(t,\cdot)-\boldsymbol{u}_0(\cdot)\|_{L^2(\Omega)}=0.
\end{gathered}$$

Remark 3.1.2. *In the formulation above all the test-velocities are divergence-free and as a consequence the pressure term vanishes. In this section we will carry out the analysis for the velocity and stress variables only. It is known that even in the Newtonian case the pressure is only a distribution in time, when working with a no-slip boundary condition (see e.g. [Gal11]). An integrable pressure can be obtained if Navier's slip boundary condition is used instead [BGMŚ12], but in this chapter we will confine ourselves to the more common no-slip boundary condition. The same analysis carries over to the problem with Navier's boundary condition; this will be shown when dealing with the anisothermal problem in Chapter 5, where the existence of an integrable pressure is essential.*

From (2.10) we have that

$$\boldsymbol{u}\in C([0,T];(W^{1,\check{r}'}_{0,\mathrm{div}}(\Omega)^d)^*)\hookrightarrow C_w([0,T];(W^{1,\check{r}'}_{0,\mathrm{div}}(\Omega)^d)^*),$$

and since $\check{r}\leq r'$ we also know that $L^2_{\mathrm{div}}(\Omega)^d\hookrightarrow(W^{1,\check{r}'}_{0,\mathrm{div}}(\Omega)^d)^*$. Combined with (2.11) this yields $\boldsymbol{u}\in C_w([0,T];L^2_{\mathrm{div}}(\Omega)^d)$ and hence the initial condition only makes sense a priori in this weaker sense. However, for this problem it will be proved that it also holds in the stronger sense described above.

For a given time step τ_m and $j\in\{1,\dots,T/\tau_m\}$, let us define $\boldsymbol{f}_j\in W^{-1,r'}(\Omega)^d$ and $\boldsymbol{\mathcal{D}}^k_j\colon\Omega\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\to\mathbb{R}^{d\times d}_{\mathrm{sym}}$ as the time averages associated with $\boldsymbol{f}$ and $\boldsymbol{\mathcal{D}}^k$, respectively (recall (2.15) and (2.26)). The time derivative will be discretised using an implicit Euler scheme; higher order time stepping techniques might not be advantageous here because higher regularity in time of weak solutions to the problem is not guaranteed a priori. The discrete formulation of the problem can now be introduced.

Formulation $\check{\mathbf{A}}_{\mathbf{k,n,m,l}}$. For $j \in \{1, \dots, T/\tau_m\}$, find functions $\mathbf{S}_j^{k,n,m,l} \in \Sigma^n$ and $\boldsymbol{u}_j^{k,n,m,l} \in V^n_{\mathrm{div}}$ such that:

$$
\begin{aligned}
\int_\Omega (\boldsymbol{\mathcal{D}}_j^k(\cdot, \mathbf{S}_j^{k,n,m,l}) - \mathbf{D}(\boldsymbol{u}_j^{k,n,m,l})) : \boldsymbol{\tau} = 0 \qquad\qquad\qquad\qquad & \forall\, \boldsymbol{\tau} \in \Sigma^n, \\
\frac{1}{\tau_m} \int_\Omega (\boldsymbol{u}_j^{k,n,m,l} - \boldsymbol{u}_{j-1}^{k,n,m,l}) \cdot \boldsymbol{v} + \frac{1}{l} \int_\Omega |\boldsymbol{u}_j^{k,n,m,l}|^{2r'-2} \boldsymbol{u}_j^{k,n,m,l} \cdot \boldsymbol{v} \qquad & \\
+ \int_\Omega (\mathbf{S}_j^{k,n,m,l} : \mathbf{D}(\boldsymbol{v}) + \mathcal{B}(\boldsymbol{u}_j^{k,n,m,l}, \boldsymbol{u}_j^{k,n,m,l}, \boldsymbol{v})) = \langle \boldsymbol{f}_j, \boldsymbol{v} \rangle \quad & \forall\, \boldsymbol{v} \in V^n_{\mathrm{div}}, \\
\boldsymbol{u}_0^{k,n,m,l} = P^n_{\mathrm{div}} \boldsymbol{u}_0. &
\end{aligned}
$$

Here $P^n_{\mathrm{div}} \colon L^2(\Omega)^d \to V^n_{\mathrm{div}}$ is simply the L^2–projection defined through

$$
\int_\Omega P^n_{\mathrm{div}} \boldsymbol{v} \cdot \boldsymbol{w} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{w} \qquad \forall\, \boldsymbol{w} \in V^n_{\mathrm{div}}. \tag{3.3}
$$

The form $\mathcal{B}$ is meant to represent the convective term and is defined for functions $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in C_0^\infty(\Omega)^d$ as:

$$
\mathcal{B}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := \begin{cases} -\displaystyle\int_\Omega \boldsymbol{u} \otimes \boldsymbol{v} : \mathbf{D}(\boldsymbol{w}), & \text{if } V^n_{\mathrm{div}} \subset W^{1,r}_{0,\mathrm{div}}(\Omega)^d, \\ \displaystyle\frac{1}{2} \int_\Omega \boldsymbol{u} \otimes \boldsymbol{w} : \mathbf{D}(\boldsymbol{v}) - \boldsymbol{u} \otimes \boldsymbol{v} : \mathbf{D}(\boldsymbol{w}), & \text{otherwise.} \end{cases}
$$

This definition guarantees that $\mathcal{B}(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v}) = 0$ for every $\boldsymbol{v}$ for which this expression is well defined, regardless of whether $\boldsymbol{v}$ is pointwise divergence-free or not, which is very useful when obtaining a priori estimates; it reduces to the usual weak form of the convective term whenever the velocities are exactly divergence-free. It is now necessary to check that $\mathcal{B}$ can be continuously extended to the spaces involving time. By standard function space interpolation, we have that for almost every $t \in (0, T)$:

$$
\begin{aligned}
\int_\Omega |\boldsymbol{u}(t,\cdot) \otimes \boldsymbol{v}(t,\cdot) : \mathbf{D}(\boldsymbol{w}(t,\cdot))| &\leq \|\boldsymbol{u}(t,\cdot)\|_{L^{2\tilde{r}}(\Omega)} \|\boldsymbol{v}(t,\cdot)\|_{L^{2\tilde{r}}(\Omega)} \|\mathbf{D}(\boldsymbol{w}(t,\cdot))\|_{L^{\tilde{r}'}(\Omega)} \\
&\leq \|\boldsymbol{u}(t,\cdot)\|_{L^{\frac{r(d+2)}{d}}(\Omega)} \|\boldsymbol{v}(t,\cdot)\|_{L^{\frac{r(d+2)}{d}}(\Omega)} \|\mathbf{D}(\boldsymbol{w}(t,\cdot))\|_{L^{\tilde{r}'}(\Omega)} \\
&\leq c \|\boldsymbol{u}(t,\cdot)\|_{W^{1,r}(\Omega)} \|\boldsymbol{v}(t,\cdot)\|_{W^{1,r}(\Omega)} \|\boldsymbol{w}(t,\cdot)\|_{W^{1,\tilde{r}'}(\Omega)}.
\end{aligned}
$$

As in the steady case (cf. [DKS13]), a more restrictive condition is needed in order to bound the additional term in $\mathcal{B}$ whenever the elements are not exactly divergence-free.

Namely, if we assume that $r \geq \frac{2(d+1)}{d+2}$ (this is the analogue of the condition $r \geq \frac{2d}{d+1}$ in the steady case) then there is a $q \in (1,\infty]$ such that $\frac{1}{r} + \frac{d}{r(d+2)} + \frac{1}{q} = 1$, and therefore

$$\begin{aligned}\int_\Omega |\boldsymbol{u}(t,\cdot) \otimes \boldsymbol{w}(t,\cdot) : \mathbf{D}(\boldsymbol{v}(t,\cdot))| &\leq \|\boldsymbol{u}(t,\cdot)\|_{L^{\frac{r(d+2)}{d}}(\Omega)} \|\mathbf{D}(\boldsymbol{v}(t,\cdot))\|_{L^r(\Omega)} \|\boldsymbol{w}(t,\cdot)\|_{L^q(\Omega)} \\ &\leq c\|\boldsymbol{u}(t,\cdot)\|_{W^{1,r}(\Omega)} \|\boldsymbol{v}(t,\cdot)\|_{W^{1,r}(\Omega)} \|\boldsymbol{w}(t,\cdot)\|_{W^{1,\tilde{r}'}(\Omega)}.\end{aligned}$$

On the other hand, using Hölder's inequality we can also obtain the estimate

$$\begin{aligned}\|\mathcal{B}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\|_{L^1(0,T)} \leq{}& \|\boldsymbol{u}\|_{L^{2r'}(Q)} \|\boldsymbol{v}\|_{L^{2r'}(Q)} \|\boldsymbol{w}\|_{L^r(0,T;W^{1,r}(\Omega))} \\ &+ \|\boldsymbol{u}\|_{L^{2r'}(Q)} \|\boldsymbol{w}\|_{L^{2r'}(Q)} \|\boldsymbol{v}\|_{L^r(0,T;W^{1,r}(\Omega))},\end{aligned}$$

which means that if the $L^{2r'}(Q)^d$ norm of $\boldsymbol{u}$ is finite, then the additional restriction $r \geq \frac{2(d+1)}{d+2}$ is not needed. Moreover, this would also imply that the velocity is an admissible test function, which is useful in the convergence analysis. This motivates the introduction of the penalty term in Formulation $\check{\mathrm{A}}_{\mathrm{k,n,m,l}}$. This formulation is a four-step approximation in which the indices k, n, m, l refer to the approximation of the graph by smooth functions, the finite element discretisation, the discretisation in time, and the penalty term, respectively.

While Formulation $\check{\mathrm{A}}_{\mathrm{k,n,m,l}}$ does not contain the pressure, in practice the incompressibility condition is enforced through the addition of a Lagrange multiplier $p_j^{k,n,m,l} \in M_0^n$, which could be thought of as the pressure in the system (the reason for the omission of the pressure in the analysis is explained in Remark 3.1.2). For this reason it is necessary to consider additional assumptions that guarantee inf-sup stability of the spaces V^n and M^n (see Assumptions 2.5.3 and 2.5.4). In case the problem does have an integrable pressure p, then it is expected that the sequence of discrete pressures converges to it in $L^1(Q)$.

Remark 3.1.3. *Assumption (A5) also implies the existence of a selection* $\boldsymbol{\mathcal{S}} \colon Q \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *such that* $(\boldsymbol{\tau}, \boldsymbol{\mathcal{S}}(z,\boldsymbol{\tau})) \in \mathcal{A}(z)$ *for all* $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$*, and some models can be written more naturally with a selection of this form; the same analysis as the one presented in this work can be applied to that situation. In fact, in practice it is not necessary to find a selection in order to perform the computations, i.e. in the simulations it is possible to work directly with the implicit function* $\mathbf{G}$.

Remark 3.1.4. *In this work we did not consider a dual formulation, e.g. where we seek* $\mathbf{S}$ *in* $H(\mathrm{div};\Omega)$*-type spaces, because for the unsteady problem we do not have at our disposal results that guarantee the integrability of* $\operatorname{div}\mathbf{S}$.

In the next theorem, convergence of the sequence of discrete solutions to a weak solution of the problem is proved. Since the ideas and arguments contained in the proof follow a similar approach to the one presented in [ST19], we will not include here all the details of the calculations unless there is a significant difference.

Theorem 3.1.5. *Assume that $r > \frac{2d}{d+2}$, and let $\{\Sigma^n, V^n, M^n\}_{n\in\mathbb{N}}$ be a family of finite element spaces satisfying Assumptions 2.5.1–2.5.3. Then for $k,n,m,l \in \mathbb{N}$ there exists a sequence $\{(\mathbf{S}_j^{k,n,m,l}, \boldsymbol{u}_j^{k,n,m,l})\}_{j=1}^{T/\tau_m}$ of solutions of Formulation $\check{\mathrm{A}}_{\mathrm{k,n,m,l}}$, and a couple $(\mathbf{S},\boldsymbol{u}) \in L^{r'}_{\mathrm{sym,tr}}(Q)^{d\times d} \times L^r(0,T;W^{1,r}_{0,\mathrm{div}}(\Omega)^d) \cap L^\infty(0,T;L^2_{\mathrm{div}}(\Omega)^d)$ such that the corresponding time interpolants (recall (2.24) and (2.25)) $\overline{\boldsymbol{u}}^{k,n,m,l}$, $\tilde{\boldsymbol{u}}^{k,n,m,l}$ and $\overline{\mathbf{S}}^{k,n,m,l}$ satisfy (up to a subsequence):*

$$\begin{aligned} \overline{\mathbf{S}}^{k,n,m,l} &\rightharpoonup \mathbf{S} && \text{weakly in } L^{r'}(Q)^{d\times d}, \\ \overline{\boldsymbol{u}}^{k,n,m,l} &\rightharpoonup \boldsymbol{u} && \text{weakly in } L^r(0,T;W^{1,r}_0(\Omega)^d), \\ \overline{\boldsymbol{u}}^{k,n,m,l}, \tilde{\boldsymbol{u}}^{k,n,m,l} &\overset{*}{\rightharpoonup} \boldsymbol{u} && \text{weakly* in } L^\infty(0,T;L^2(\Omega)^d), \end{aligned} \tag{3.4}$$

and $(\mathbf{S},\boldsymbol{u})$ solves Formulation $\check{\mathrm{A}}$, with the limits taken in the order $k\to\infty$, $(n,m)\to\infty$ and $l\to\infty$.

Proof. The idea of the proof is common in the analysis of nonlinear PDE: we obtain a priori estimates and use compactness arguments to pass to the limit in the equation. In order to prove the existence of solutions of Formulation $\check{\mathrm{A}}_{\mathrm{k,n,m,l}}$, we need to check that given $(\mathbf{S}_{j-1}^{k,n,m,l}, \boldsymbol{u}_{j-1}^{k,n,m,l})$, we can find $(\mathbf{S}_j^{k,n,m,l}, \boldsymbol{u}_j^{k,n,m,l})$, for $j\in\{1,\dots,T/\tau_m\}$. Testing the equation with $(\mathbf{S}_j^{k,n,m,l}, \boldsymbol{u}_j^{k,n,m,l})$, we see that:

$$\int_\Omega \boldsymbol{\mathcal{D}}^k(\cdot,\mathbf{S}_j^{k,n,m,l}):\mathbf{S}_j^{k,n,m,l} + \frac{1}{l}\|\boldsymbol{u}_j^{k,n,m,l}\|^{2r'}_{L^{2r'}(\Omega)} \leq \langle \boldsymbol{f}, \boldsymbol{u}_j^{k,n,m,l}\rangle + \frac{1}{\tau_m}\int_\Omega \boldsymbol{u}_{j-1}^{k,n,m,l}\cdot\boldsymbol{u}_j^{k,n,m,l}. \tag{3.5}$$

On the other hand, since all norms are equivalent in a finite-dimensional normed linear space, there is a constant $c_n > 0$ such that:

$$\|\boldsymbol{v}\|_{W^{1,r}(\Omega)} \leq c_n \|\boldsymbol{v}\|_{L^{2r'}(\Omega)} \qquad \forall\, \boldsymbol{v}\in V^n_{\mathrm{div}}. \tag{3.6}$$

The constant c_n may blow up as $n\to\infty$, but since n is fixed for now this does not pose a problem. Now, recalling (2.17) and combining (3.5) and (3.6) with a standard corollary of Brouwer's Fixed Point Theorem (cf. [GR86, ch. IV, Cor. 1.1]) we obtain the existence of solutions of Formulation $\check{\mathrm{A}}_{\mathrm{k,n,m,l}}$. In the first time step (i.e. $j=1$), it is essential to use the fact that the projection P^n_{div} is stable:

$$\|P^n_{\mathrm{div}}\boldsymbol{u}_0\|_{L^2(\Omega)} \leq \|\boldsymbol{u}_0\|_{L^2(\Omega)}. \tag{3.7}$$

The estimate (3.6) suffices to guarantee the existence of discrete solutions, but in order to pass to the limit $n \to \infty$, an estimate that does not degenerate as $n \to \infty$ is required. This uniform estimate is a consequence of the discrete inf-sup condition (2.23):

$$\gamma_r \|\boldsymbol{u}_j^{k,n,m,l}\|_{W^{1,r}(\Omega)} \leq \|\boldsymbol{\mathcal{D}}_j^k(\cdot, \mathbf{S}_j^{k,n,m,l})\|_{L^r(\Omega)}. \tag{3.8}$$

Therefore, the following a priori estimate holds:

$$\begin{aligned}
&\sup_{j\in\{1,\dots,T/\tau_m\}} \|\boldsymbol{u}_j^{k,n,m,l}\|_{L^2(\Omega)}^2 + \sum_{j=1}^{T/\tau_m} \|\boldsymbol{u}_j^{k,n,m,l} - \boldsymbol{u}_{j-1}^{k,n,m,l}\|_{L^2(\Omega)}^2 \\
&+ \tau_m \sum_{j=1}^{T/\tau_m} \|\mathbf{S}_j^{k,n,m,l}\|_{L^{r'}(\Omega)}^{r'} + \tau_m \sum_{j=1}^{T/\tau_m} \|\boldsymbol{u}_j^{k,n,m,l}\|_{W^{1,r}(\Omega)}^r \\
&+ \sum_{j=1}^{T/\tau_m} \|\boldsymbol{\mathcal{D}}^k(\cdot,\cdot,\mathbf{S}_j^{k,n,m,l})\|_{L^r(Q_{j-1}^j)}^r + \frac{\tau_m}{l} \sum_{j=1}^{T/\tau_m} \|\boldsymbol{u}_j^{k,n,m,l}\|_{L^{2r'}(\Omega)}^{2r'} \leq c,
\end{aligned} \tag{3.9}$$

where c is a positive constant that depends on the data; in particular, c is independent of k, n, m and l. Let $\overline{\boldsymbol{u}}^{k,n,m,l} \in L^\infty(0,T;V_{\text{div}}^n)$ and $\tilde{\boldsymbol{u}}^{k,n,m,l} \in C([0,T];V_{\text{div}}^n)$ be the piecewise constant and piecewise linear interpolants defined by the sequence $\{\boldsymbol{u}_j^{k,n,m,l}\}_{j=1}^{T/\tau_m}$ (see (2.24) and (2.25)) and let $\overline{\mathbf{S}}^{k,n,m,l} \in L^\infty(0,T;\Sigma^n)$ be the piecewise constant interpolant defined by the sequence $\{\mathbf{S}_j^{k,n,m,l}\}_{j=1}^{T/\tau_m}$. Furthermore, define also the piecewise constant interpolants:

$$\overline{\boldsymbol{f}}(t,\cdot) := \boldsymbol{f}_j(\cdot), \qquad \overline{\boldsymbol{\mathcal{D}}}^k(t,\cdot,\cdot) := \boldsymbol{\mathcal{D}}_j^k(\cdot,\cdot), \quad t \in (t_{j-1},t_j], \quad j \in \{1,\dots,T/\tau_m\}$$

Then the discrete formulation can be rewritten as:

$$\begin{aligned}
&\int_\Omega (\overline{\boldsymbol{\mathcal{D}}}^k(t,\cdot,\overline{\mathbf{S}}^{k,n,m,l}) - \mathbf{D}(\overline{\boldsymbol{u}}^{k,n,m,l})) : \boldsymbol{\tau} = 0 && \forall\, \boldsymbol{\tau} \in \Sigma_{\text{sym}}^n, \\
&\int_\Omega \partial_t \tilde{\boldsymbol{u}}^{k,n,m,l} \cdot \boldsymbol{v} + \frac{1}{l}\int_\Omega |\overline{\boldsymbol{u}}^{k,n,m,l}|^{2r'-2}\overline{\boldsymbol{u}}^{k,n,m,l} \cdot \boldsymbol{v} \\
&\qquad + \int_\Omega (\overline{\mathbf{S}}^{k,n,m,l} : \mathbf{D}(\boldsymbol{v}) + \mathcal{B}(\overline{\boldsymbol{u}}^{k,n,m,l}, \overline{\boldsymbol{u}}^{k,n,m,l}, \boldsymbol{v})) = \langle \overline{\boldsymbol{f}}, \boldsymbol{v}\rangle && \forall\, \boldsymbol{v} \in V_{\text{div}}^n, \\
&\tilde{\boldsymbol{u}}^{k,n,m,l}(0,\cdot) = P_{\text{div}}^n \boldsymbol{u}_0(\cdot).
\end{aligned}$$

The a priori estimate (3.9) can in turn be written as:

$$\begin{aligned}
&\|\overline{\boldsymbol{u}}^{k,n,m,l}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \tau_m \|\partial_t \tilde{\boldsymbol{u}}^{k,n,m,l}\|_{L^2(Q)}^2 + \|\overline{\mathbf{S}}^{k,n,m,l}\|_{L^{r'}(Q)}^{r'} \\
&+ \|\overline{\boldsymbol{u}}^{k,n,m,l}\|_{L^r(0,T;W^{1,r}(\Omega))}^r + \|\boldsymbol{\mathcal{D}}^k(\cdot,\cdot,\overline{\mathbf{S}}^{k,n,m,l})\|_{L^r(Q)}^r + \frac{1}{l}\|\overline{\boldsymbol{u}}^{k,n,m,l}\|_{L^{2r'}(Q)}^{2r'} \leq c.
\end{aligned} \tag{3.10}$$

Using the equivalence of norms in finite-dimensional spaces we also obtain

$$\|\partial_t \tilde{\boldsymbol{u}}^{k,n,m,l}\|_{L^\infty(0,T;L^2(\Omega))} \leq c(n)\|\partial_t \tilde{\boldsymbol{u}}^{k,n,m,l}\|_{L^2(Q)},$$

and together with the a priori estimate this implies that

$$\|\tilde{\boldsymbol{u}}^{k,n,m,l}\|_{W^{1,\infty}(0,T;L^2(\Omega))} \leq c(n,m). \tag{3.11}$$

Therefore, up to subsequences, as $k \to \infty$ we have:

$$\begin{aligned}
\overline{\boldsymbol{u}}^{k,n,m,l} &\to \overline{\boldsymbol{u}}^{n,m,l} && \text{strongly in } L^\infty(0,T;L^2(\Omega)^d),\\
\tilde{\boldsymbol{u}}^{k,n,m,l} &\to \tilde{\boldsymbol{u}}^{n,m,l} && \text{strongly in } W^{1,\infty}(0,T;L^2(\Omega)^d),\\
\overline{\boldsymbol{u}}^{k,n,m,l} &\to \overline{\boldsymbol{u}}^{n,m,l} && \text{strongly in } L^{2r'}(Q)^d,\\
\overline{\boldsymbol{u}}^{k,n,m,l} &\to \overline{\boldsymbol{u}}^{n,m,l} && \text{strongly in } L^r(0,T;W^{1,r}_0(\Omega)^d),\\
\overline{\mathbf{S}}^{k,n,m,l} &\to \overline{\mathbf{S}}^{n,m,l} && \text{strongly in } L^{r'}(Q)^{d\times d},\\
\boldsymbol{\mathcal{D}}^k(\cdot,\cdot,\overline{\mathbf{S}}^{k,n,m,l}) &\rightharpoonup \mathbf{D}^{n,m,l} && \text{weakly in } L^r(Q)^{d\times d},\\
\overline{\boldsymbol{\mathcal{D}}}^k(\cdot,\cdot,\overline{\mathbf{S}}^{k,n,m,l}) &\rightharpoonup \overline{\mathbf{D}}^{n,m,l} && \text{weakly in } L^r(Q)^{d\times d},\\
\boldsymbol{\mathcal{D}}^k_j(\cdot,\mathbf{S}^{k,n,m,l}_j) &\rightharpoonup \mathbf{D}^{n,m,l}_j && \text{weakly in } L^r(\Omega)^{d\times d}, \text{ for } j\in\{1,\dots,T/\tau_m\}.
\end{aligned}$$

Since the function $\mathbf{D}^k_j$ is simply an average in time, the uniqueness of the weak limit implies that

$$\mathbf{D}^{n,m,l}_j(\cdot) = \frac{1}{\tau_m}\int_{t_{j-1}}^{t_j} \mathbf{D}^{n,m,l}(t,\cdot)\,\mathrm{d}t, \qquad j\in\{1,\dots,T/\tau_m\}, \tag{3.12}$$

and that $\overline{\mathbf{D}}^{n,m,l}$ is the piecewise constant interpolant determined by the sequence $\{\mathbf{D}^{n,m,l}_j\}_{j=1}^{T/\tau_m}$. Moreover, since the convergence of the velocity and stress sequences is strong, it is straightforward to pass to the limit $k\to\infty$ and thus we obtain

$$\begin{aligned}
&\int_\Omega (\overline{\mathbf{D}}^{n,m,l} - \mathbf{D}(\overline{\boldsymbol{u}}^{n,m,l})) : \boldsymbol{\tau} = 0 && \forall\, \boldsymbol{\tau}\in\Sigma^n,\\
&\int_\Omega \partial_t \tilde{\boldsymbol{u}}^{n,m,l}\cdot\boldsymbol{v} + \frac{1}{l}\int_\Omega |\overline{\boldsymbol{u}}^{n,m,l}|^{2r'-2}\,\overline{\boldsymbol{u}}^{n,m,l}\cdot\boldsymbol{v}\\
&\qquad + \int_\Omega (\overline{\mathbf{S}}^{n,m,l} : \mathbf{D}(\boldsymbol{v}) + \mathcal{B}(\overline{\boldsymbol{u}}^{n,m,l},\overline{\boldsymbol{u}}^{n,m,l},\boldsymbol{v})) = \langle \overline{\boldsymbol{f}},\boldsymbol{v}\rangle && \forall\,\boldsymbol{v}\in V^n_{\mathrm{div}}.
\end{aligned}$$

It is also clear that the initial condition $\tilde{\boldsymbol{u}}^{n,m,l}(0,\cdot) = P^n_{\mathrm{div}}\boldsymbol{u}_0(\cdot)$ holds, since the expression on the right-hand side is independent of k. The identification of the constitutive relation can be carried out using (2.18) in exactly the same manner as in [ST19], which means that (the strong convergence is again essential):

$$(\mathbf{D}^{n,m,l},\overline{\mathbf{S}}^{n,m,l}) \in \mathcal{A}(\cdot), \text{ a.e. in } (0,T)\times\Omega. \tag{3.13}$$

The next step is to take the limit in both the time and space discretisations simultaneously. The weak lower semicontinuity of the norms and the estimate (3.10) imply that:

$$\begin{aligned} &\|\overline{\boldsymbol{u}}^{n,m,l}\|^2_{L^\infty(0,T;L^2(\Omega))} + \tau_m \|\partial_t \tilde{\boldsymbol{u}}^{n,m,l}\|^2_{L^2(Q)} + \|\overline{\mathbf{S}}^{n,m,l}\|^{r'}_{L^{r'}(Q)} \\ &+ \|\overline{\boldsymbol{u}}^{n,m,l}\|^r_{L^r(0,T;W^{1,r}(\Omega))} + \|\mathbf{D}^{n,m,l}\|^r_{L^r(Q)} + \frac{1}{l}\|\overline{\boldsymbol{u}}^{n,m,l}\|^{2r'}_{L^{2r'}(Q)} \leq c, \end{aligned} \tag{3.14}$$

and

$$\|\tilde{\boldsymbol{u}}^{n,m,l}\|^2_{L^\infty(0,T;L^2(\Omega))} = \|\overline{\boldsymbol{u}}^{n,m,l}\|^2_{L^\infty(0,T;L^2(\Omega))} \leq c, \tag{3.15}$$

where c is a constant, independent of n, m and l. Consequently, there exist (not relabelled) subsequences such that, as $n, m \to \infty$:

$$\begin{aligned} \overline{\boldsymbol{u}}^{n,m,l} &\overset{*}{\rightharpoonup} \boldsymbol{u}^l && \text{weakly* in } L^\infty(0,T;L^2(\Omega)^d), \\ \tilde{\boldsymbol{u}}^{n,m,l} &\overset{*}{\rightharpoonup} \boldsymbol{u}^l && \text{weakly* in } L^\infty(0,T;L^2(\Omega)^d), \\ \overline{\boldsymbol{u}}^{n,m,l} &\rightharpoonup \boldsymbol{u}^l && \text{weakly in } L^r(0,T;W^{1,r}_0(\Omega)^d), \\ \overline{\mathbf{S}}^{n,m,l} &\rightharpoonup \mathbf{S}^l && \text{weakly in } L^{r'}(Q)^{d\times d}, \\ \mathbf{D}^{n,m,l} &\rightharpoonup \mathbf{D}^l && \text{weakly in } L^r(Q)^{d\times d}, \\ \overline{\mathbf{D}}^{n,m,l} &\rightharpoonup \overline{\mathbf{D}}^l && \text{weakly in } L^r(Q)^{d\times d}, \\ \frac{1}{l}|\overline{\boldsymbol{u}}^{n,m,l}|^{2r'-2}\overline{\boldsymbol{u}}^{n,m,l} &\rightharpoonup \frac{1}{l}|\boldsymbol{u}^l|^{2r'-2}\boldsymbol{u}^{n,m,l} && \text{weakly in } L^{(2r')'}(Q)^d. \end{aligned}$$

At this point it is a standard step to use the Aubin–Lions lemma to obtain strong convergence of subsequences. However, following [ST19], we will instead use Simon's compactness lemma; this choice is made to avoid the need for stability estimates of P^n_{div} in Sobolev norms, which would require additional assumptions on the mesh. To apply this lemma, it will be more convenient to work with the modified interpolant:

$$\hat{\boldsymbol{u}}^{n,m,l}(t,\cdot) := \begin{cases} \boldsymbol{u}^{n,m,l}_1(\cdot), & \text{if } t \in [0,t_1), \\ \tilde{\boldsymbol{u}}^{n,m,l}(t,\cdot), & \text{if } t \in [t_1,T]. \end{cases}$$

Let $\epsilon > 0$ be such that $s+\epsilon < T$ and let $\boldsymbol{v} \in V^n_{\mathrm{div}}$. Then, using the definition of $\hat{\boldsymbol{u}}^{n,m,l}$ we have

$$\begin{aligned} &\int_\Omega (\hat{\boldsymbol{u}}^{n,m,l}(s+\epsilon,x) - \hat{\boldsymbol{u}}^{n,m,l}(s,x)) \cdot \boldsymbol{v}(x)\,\mathrm{d}x \\ &= \int_{\max(s,\tau_m)}^{s+\epsilon} \int_\Omega \partial_t \hat{\boldsymbol{u}}^{n,m,l}(t,x) \cdot \boldsymbol{v}(x)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\max(s,\tau_m)}^{s+\epsilon} \int_\Omega \partial_t \tilde{\boldsymbol{u}}^{n,m,l}(t,x) \cdot \boldsymbol{v}(x)\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= \int_{\max(s,\tau_m)}^{s+\epsilon} \Bigg(-\frac{1}{l} \int_\Omega |\overline{\boldsymbol{u}}^{n,m,l}(t,x)|^{2r'-2} \overline{\boldsymbol{u}}^{n,m,l}(t,x) \cdot \boldsymbol{v}(x) \,\mathrm{d}x \\
&- \int_\Omega (\overline{\mathbf{S}}^{n,m,l}(t,x) : \mathbf{D}(\boldsymbol{v}(x)) + \mathcal{B}(\overline{\boldsymbol{u}}^{n,m,l}(t,x), \overline{\boldsymbol{u}}^{n,m,l}(t,x), \boldsymbol{v}(x))) \,\mathrm{d}x - \langle \overline{\boldsymbol{f}}(t), \boldsymbol{v} \rangle \Bigg) \,\mathrm{d}t \\
&\le c(l) \left(\left(\int_{\max(s,\tau_m)}^{s+\epsilon} \|\boldsymbol{v}\|_{W^{1,r}(\Omega)}^r \,\mathrm{d}t \right)^{1/r} + \left(\int_{\max(s,\tau_m)}^{s+\epsilon} \|\boldsymbol{v}\|_{L^{2r'}(\Omega)}^{2r'} \,\mathrm{d}t \right)^{1/2r'} \right) \\
&\le c(l)(\epsilon^{1/r} + \epsilon^{1/2r'}) \left(\|\boldsymbol{v}\|_{W^{1,r}(\Omega)} + \|\boldsymbol{v}\|_{L^{2r'}(\Omega)} \right).
\end{aligned}
$$

Choosing $\boldsymbol{v} = \hat{\boldsymbol{u}}^{n,m,l}(s+\epsilon,\cdot) - \hat{\boldsymbol{u}}^{n,m,l}(s,\cdot)$ we conclude that

$$
\int_0^{T-\epsilon} \|\hat{\boldsymbol{u}}^{n,m,l}(s+\epsilon,\cdot) - \hat{\boldsymbol{u}}^{n,m,l}(s,\cdot)\|_{L^2(\Omega)}^2 \,\mathrm{d}s \to 0, \text{ as } \epsilon \to 0.
$$

On the other hand, the a priori estimates imply that $\hat{\boldsymbol{u}}^{n,m,l}$ is bounded (uniformly in $n, m \in \mathbb{N}$) in $L^2(Q)^d$ and $L^1(0,T;W_0^{1,r}(\Omega)^d)$. Moreover, since $r > \frac{2d}{d+2}$, the embedding $W^{1,r}(\Omega)^d \hookrightarrow L^2(\Omega)^d$ is compact and thus Simon's compactness lemma guarantees the strong convergence:

$$
\hat{\boldsymbol{u}}^{n,m,l} \to \boldsymbol{u}^l \qquad \text{strongly in } L^2(Q)^d. \tag{3.16}
$$

Since the interpolants converge to the same limit as $\tau_m \to 0$, using standard function space interpolation (and recalling (2.9)) we also obtain that, as $n, m \to \infty$:

$$
\begin{aligned}
\tilde{\boldsymbol{u}}^{n,m,l} &\to \boldsymbol{u}^l && \text{strongly in } L^p(0,T;L^2(\Omega)^d), && \text{(3.17a)} \\
\overline{\boldsymbol{u}}^{n,m,l} &\to \boldsymbol{u}^l && \text{strongly in } L^p(0,T;L^2(\Omega)^d) \cap L^q(Q), && \text{(3.17b)}
\end{aligned}
$$

for $p \in [1,\infty)$ and $q \in [1, \max(2r', \frac{q(d+2)}{d}))$.

Now, using the property (2.21), we can check that $\boldsymbol{u}^l$ is actually divergence-free:

$$
0 = \int_0^T \int_\Omega \phi\, \Pi_M^n q \,\mathrm{div} \overline{\boldsymbol{u}}^{n,m,l} \to \int_0^T \int_\Omega \phi\, q \,\mathrm{div} \boldsymbol{u}^l \quad \forall\, q \in L^{r'}(\Omega),\ \phi \in C_0^\infty(0,T). \tag{3.18}
$$

Furthermore, (2.21) also yields convergence of the initial condition, as $n, m \to \infty$:

$$
\tilde{\boldsymbol{u}}^{n,m,l}(0,\cdot) = P_{\mathrm{div}}^n \boldsymbol{u}_0 \to \boldsymbol{u}_0 \qquad \text{strongly in } L^2(\Omega)^d. \tag{3.19}
$$

The functions $\mathbf{D}^l$ and $\overline{\mathbf{D}}^l$ can easily be identified using the property (2.28) and the definition of the piecewise constant interpolant (3.12). Indeed, for an arbitrary $\boldsymbol{\sigma} \in C_0^\infty(Q)^{d\times d}$ we have, as $n, m \to \infty$:

$$
\int_0^T \int_\Omega \overline{\mathbf{D}}^{n,m,l} : \boldsymbol{\sigma} = \int_0^T \int_\Omega \mathbf{D}^{n,m,l} : \overline{\boldsymbol{\sigma}} \to \int_0^T \int_\Omega \mathbf{D}^l : \boldsymbol{\sigma}. \tag{3.20}
$$

The uniqueness of the weak limit then implies that $\mathbf{D}^l = \overline{\mathbf{D}}^l$.

Combining all these properties and using an analogous computation to (3.18) it is possible to prove that the limiting functions are a solution of the following problem:

$$\int_0^T \int_\Omega (\mathbf{D}^l - \mathbf{D}(\boldsymbol{u}^l)) : \boldsymbol{\tau}\, \varphi = 0 \qquad \forall\, \boldsymbol{\tau} \in C^\infty_{0,\mathrm{sym}}(\Omega)^{d\times d},\ \varphi \in C^\infty_0(0,T),$$

$$-\int_0^T \int_\Omega \boldsymbol{u}^l \cdot \boldsymbol{v}\, \partial_t \varphi - \int_\Omega \boldsymbol{u}_0 \cdot \boldsymbol{v} \varphi(0) + \int_0^T \int_\Omega (\mathbf{S}^l - \boldsymbol{u}^l \otimes \boldsymbol{u}^l) : \mathbf{D}(\boldsymbol{v})\, \varphi$$
$$+\frac{1}{l} \int_0^T \int_\Omega |\boldsymbol{u}^l|^{2r'-2} \boldsymbol{u}^l \cdot \boldsymbol{v}\, \varphi = \int_0^T \langle \boldsymbol{f}, \boldsymbol{v} \rangle\, \varphi \qquad \forall\, \boldsymbol{v} \in C^\infty_{0,\mathrm{div}}(\Omega)^d,\ \varphi \in C^\infty_0(-T,T).$$

From the equation above and the estimate (2.9) we then see that the distributional time derivative belongs to the spaces:

$$\partial_t \boldsymbol{u}^l \in L^{\min(r',(2r')')}(0,T; (W^{1,r}_{0,\mathrm{div}}(\Omega)^d \cap L^{2r'}(\Omega)^d)^*), \tag{3.21a}$$
$$\partial_t \boldsymbol{u}^l \in L^{\min(\tilde{r},(2r')')}(0,T; (W^{1,\tilde{r}'}_{0,\mathrm{div}}(\Omega)^d)^*). \tag{3.21b}$$

It is important to note that (3.21b) holds uniformly in $l \in \mathbb{N}$, while (3.21a) does not. Now, observe that

$$W^{1,r}_{0,\mathrm{div}}(\Omega)^d \cap L^{2r'}(\Omega)^d \hookrightarrow L^2_{\mathrm{div}}(\Omega)^d \hookrightarrow (L^2_{\mathrm{div}}(\Omega)^d)^* \hookrightarrow (W^{1,r}_{0,\mathrm{div}}(\Omega)^d \cap L^{2r'}(\Omega)^d)^*.$$

Combining this with (2.10), (2.11), and the fact that $\boldsymbol{u}^l \in L^\infty(0,T; L^2_{\mathrm{div}}(\Omega)^d)$ guarantees that $\boldsymbol{u}^l \in C_w([0,T], L^2_{\mathrm{div}}(\Omega)^d)$. Let $\boldsymbol{v} \in C^\infty_{0,\mathrm{div}}(\Omega)^d$ and $\varphi \in C^\infty(-T,T)$ be such that $\varphi(0) = 1$; then the following equality holds:

$$\int_0^T \int_\Omega \partial_t (\boldsymbol{u}^l \varphi) \cdot \boldsymbol{v} = -\int_\Omega \boldsymbol{u}^l(0,\cdot) \cdot \boldsymbol{v}\, \varphi(0). \tag{3.22}$$

On the other hand, using the equation we also have that:

$$\int_0^T \int_\Omega \partial_t (\boldsymbol{u}^l \varphi) \cdot \boldsymbol{v} = \int_0^T \int_\Omega \partial_t \boldsymbol{u}^l \cdot \boldsymbol{v}\, \varphi + \int_0^T \int_\Omega \boldsymbol{u}^l \cdot \boldsymbol{v}\, \partial_t \varphi = -\int_\Omega \boldsymbol{u}_0 \cdot \boldsymbol{v}\, \varphi(0). \tag{3.23}$$

Comparing (3.22) and (3.23) we conclude that $\boldsymbol{u}^l(0,\cdot) = \boldsymbol{u}_0(\cdot)$. This proves that the initial condition is attained in the weak sense expected a priori from the embeddings; however, in this case the stronger condition

$$\operatorname*{ess\,lim}_{t\to 0^+} \|\boldsymbol{u}^l(t,\cdot) - \boldsymbol{u}_0(\cdot)\|_{L^2(\Omega)} = 0, \tag{3.24}$$

holds. To see this, note that (3.17a) guarantees that, up to a subsequence, $\tilde{\boldsymbol{u}}^{n,m,l}(t,\cdot) \to \tilde{\boldsymbol{u}}^l(t,\cdot)$ in $L^2(\Omega)^d$ for almost every $t \in [0,T]$, and therefore

$$\|\boldsymbol{u}^l(t,\cdot) - \boldsymbol{u}_0(\cdot)\|^2_{L^2(\Omega)} = \limsup_{n,m\to\infty} \|\tilde{\boldsymbol{u}}^{n,m,l}(t,\cdot) - \tilde{\boldsymbol{u}}^{n,m,l}(0,\cdot)\|^2_{L^2(\Omega)}$$

$$
\begin{aligned}
&= \limsup_{n,m\to\infty} \Big(\|\tilde{\boldsymbol{u}}^{n,m,l}(t,\cdot)\|^2_{L^2(\Omega)} - \|\tilde{\boldsymbol{u}}^{n,m,l}(0,\cdot)\|^2_{L^2(\Omega)} \\
&\qquad + 2\int_\Omega (\tilde{\boldsymbol{u}}^{n,m,l}(0,\cdot) - \tilde{\boldsymbol{u}}^{n,m,l}(t,\cdot)) \cdot \tilde{\boldsymbol{u}}^{n,m,l}(0,\cdot) \Big) \\
&\le \limsup_{n,m\to\infty} \left(\int_0^t \langle \overline{\boldsymbol{f}}, \overline{\boldsymbol{u}}^{n,m,l} \rangle + 2\int_\Omega (\tilde{\boldsymbol{u}}^{n,m,l}(0,\cdot) - \tilde{\boldsymbol{u}}^{n,m,l}(t,\cdot)) \cdot \tilde{\boldsymbol{u}}^{n,m,l}(0,\cdot) \right) \\
&\le \int_0^t \langle \boldsymbol{f}, \boldsymbol{u}^l \rangle + 2\int_\Omega (\boldsymbol{u}^l(0,\cdot) - \boldsymbol{u}^l(t,\cdot)) \cdot \boldsymbol{u}^l(0,\cdot),
\end{aligned}
$$

for almost every $t \in [0,T]$. Observe also that the monotonicity of the constitutive relation was used to obtain the next to last inequality. Taking the limit $t \to 0^+$ then yields (3.24).

The identification of the constitutive relation, i.e. proving that $(\mathbf{D}^l, \mathbf{S}^l) \in \mathcal{A}(\cdot)$ almost everywhere, can be carried out with the help of Lemma 2.4.1. In order to apply the lemma, the only thing that remains to be proved, since we already know that $(\mathbf{D}^{n,m,l}, \overline{\mathbf{S}}^{n,m,l}) \in \mathcal{A}(\cdot)$ almost everywhere, is that:

$$
\limsup_{n,m\to\infty} \int_0^t \int_\Omega \overline{\mathbf{S}}^{n,m,l} : \mathbf{D}^{n,m,l} \le \int_0^t \int_\Omega \mathbf{S}^l : \mathbf{D}^l, \tag{3.25}
$$

for almost every $t \in [0,T]$; then taking $t \to T$ we obtain the result in the whole domain Q. The proof of this fact is essentially the same as in [ST19] and we will not reproduce it here. Moreover, the following energy identity holds:

$$
\frac{1}{2}\|\boldsymbol{u}^l(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \mathbf{S}^l : \mathbf{D}(\boldsymbol{u}^l) + \frac{1}{l}\int_0^t \|\boldsymbol{u}^l\|^{2r'}_{L^{2r'}(\Omega)} = \int_0^t \langle \boldsymbol{f}, \boldsymbol{u}^l \rangle + \|\boldsymbol{u}_0\|^2_{L^2(\Omega)}. \tag{3.26}
$$

In time-dependent problems obtaining an energy identity of this kind is not always possible; in this case the energy equality (3.26) can be proved, since the velocity is an admissible test function in space thanks to the fact that its $L^{2r'}$-norm is under control (some mollification is needed to overcome the low integrability in time, see [Tsc18, Lio69]).

Now, (3.14) and the weak and weak* lower semicontinuity of the norms imply that

$$
\|\boldsymbol{u}^l\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{S}^l\|^{r'}_{L^{r'}(Q)} + \|\boldsymbol{u}^l\|^r_{L^r(0,T;W^{1,r}(\Omega))} + \|\mathbf{D}^l\|^r_{L^r(Q)} + \frac{1}{l}\|\boldsymbol{u}^l\|^{2r'}_{L^{2r'}(Q)} \le c, \tag{3.27}
$$

where c is a constant independent of l. From this we see that, up to subsequences, as $l \to \infty$:

$$
\begin{aligned}
\boldsymbol{u}^l &\overset{*}{\rightharpoonup} \boldsymbol{u} && \text{weakly* in } L^\infty(0,T;L^2(\Omega)^d), \\
\boldsymbol{u}^l &\rightharpoonup \boldsymbol{u} && \text{weakly in } L^r(0,T;W^{1,r}_0(\Omega)^d),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{S}^l &\rightharpoonup \mathbf{S} && \text{weakly in } L^{r'}(Q)^{d\times d}, && (3.28)\\
\mathbf{D}^l &\rightharpoonup \mathbf{D} && \text{weakly in } L^{r}(Q)^{d\times d}, &&\\
\frac{1}{l}|\boldsymbol{u}^l|^{2r'-2}\boldsymbol{u}^l &\to \boldsymbol{0} && \text{strongly in } L^{1}(Q)^{d}. &&
\end{aligned}
$$

Furthermore, since $\check{r} \leq r'$ and $r > \frac{2d}{d+2}$, the embedding $W^{1,\check{r}'}_{0,\mathrm{div}}(\Omega)^d \hookrightarrow L^2_{\mathrm{div}}(\Omega)^d$ is compact and hence by the Aubin–Lions lemma (taking into account (3.21b)) we have the strong convergence:

$$
\boldsymbol{u}^l \to \boldsymbol{u} \qquad \text{strongly in } L^2(0,T;L^2(\Omega)^d). \tag{3.29}
$$

With the convergence properties (3.28) and (3.29) it is then possible to pass to the limit and prove that the limiting functions satisfy:

$$
\begin{aligned}
\int_\Omega (\mathbf{D}-\mathbf{D}(\boldsymbol{u})):\boldsymbol{\tau} &= 0 && \forall\, \boldsymbol{\tau} \in C^\infty_{0,\mathrm{sym}}(\Omega)^{d\times d}, \text{ a.e. } t\in(0,T),\\
\langle \partial_t \boldsymbol{u}, \boldsymbol{v}\rangle + \int_\Omega (\mathbf{S}-\boldsymbol{u}\otimes\boldsymbol{u}):\mathbf{D}(\boldsymbol{v}) &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle && \forall\, \boldsymbol{v} \in C^\infty_{0,\mathrm{div}}(\Omega)^{d}, \text{ a.e. } t\in(0,T).
\end{aligned}
$$

The same argument used to obtain (3.24) can be used here to prove that the initial condition is attained in the strong sense:

$$
\operatorname*{ess\,lim}_{t\to 0^+} \|\boldsymbol{u}(t,\cdot)-\boldsymbol{u}_0(\cdot)\|_{L^2(\Omega)} = 0. \tag{3.30}
$$

Moreover, since the penalty term vanishes in the limit $l \to \infty$, we can improve the integrability in time:

$$
\partial_t \boldsymbol{u}^l \in L^{\check{r}}(0,T;(W^{1,\check{r}'}_{0,\mathrm{div}}(\Omega)^d)^*). \tag{3.31}
$$

To show that $(\mathbf{D},\mathbf{S}) \in \mathcal{A}(\cdot)$, Lemma 2.4.1 will once again be employed. The main difficulty at this stage, just like in the previous works [DKS13, ST19], is that the velocity is no longer an admissible test function (and therefore we do not have an energy equality similar to (3.26)). The idea is now to work with Lipschitz truncations of the error $\boldsymbol{e}^l := \boldsymbol{u}^l - \boldsymbol{u}$; it should be noted however that in the present case we need to verify a number of additional hypotheses before Lemma 3.1.1 can be applied.

Note that equation (3.2) in Lemma 3.1.1 is written in divergence form. We then need to make a preliminary step and write the penalty term in this form (see [ST19]). Let $B_0 \subset\subset \Omega$ be an arbitrary ball compactly contained in Ω and let $q \in [1,(2r')')$. Then from the standard theory of elliptic operators we know that for almost every $t \in [0,T]$ there is a unique $\boldsymbol{g}^l_3(t,\cdot) \in W^{2,q}(B_0)^d \cap W^{1,q}_0(B_0)$ such that:

$$
\begin{aligned}
\int_{B_0} \nabla \boldsymbol{g}^l_3(t,\cdot):\nabla\boldsymbol{v} = \frac{1}{l}\int_{B_0} |\boldsymbol{u}^l(t,\cdot)|^{2r'-2}\boldsymbol{u}^l(t,\cdot)\cdot\boldsymbol{v} \qquad \forall\, \boldsymbol{v}\in C^\infty_{0,\mathrm{div}}(\Omega)^d,\\
\|\boldsymbol{g}^l_3(t,\cdot)\|_{W^{2,q}(B_0)} \leq c \left\| \frac{1}{l}|\boldsymbol{u}^l(t,\cdot)|^{2r'-2}\boldsymbol{u}^l(t,\cdot)\right\|_{L^q(B_0)}.
\end{aligned}
$$

This means in particular (by (3.28) and standard function space interpolation) that for a fixed time interval $I_0 \subset\subset (0,T)$ we have:

$$\boldsymbol{g}_3^l \to \mathbf{0} \qquad \text{strongly in } L^q(I_0; W^{1,q}(B_0)^d), \quad \forall\, q \in [1,(2r')'). \tag{3.32}$$

Defining $Q_0 := I_0 \times B_0$ and

$$\begin{aligned}\mathbf{G}_1^l &:= \mathbf{S}^l - \mathbf{S},\\ \mathbf{G}_2^l &:= \boldsymbol{u}^l \otimes \boldsymbol{u}^l - \boldsymbol{u} \otimes \boldsymbol{u} - \nabla \boldsymbol{g}_3^l,\end{aligned}$$

we readily see that the error $\boldsymbol{e}^l$ satisfies the equation

$$\int_{Q_0} \partial_t \boldsymbol{e}^l \cdot \boldsymbol{w} = \int_{Q_0} (\mathbf{G}_1^l + \mathbf{G}_2^l) : \nabla \boldsymbol{w} \qquad \forall\, \boldsymbol{w} \in C^\infty_{0,\mathrm{div}}(Q_0)^d. \tag{3.33}$$

Additionally, as a consequence of (3.28), (3.32) and (3.29) we also have that for any $q \in [1, \min(\check{r}, (2r')')$, the sequence $\boldsymbol{u}^l$ is bounded in $L^\infty(I_0; W^{1,q}(Q_0)^d)$ and that:

$$\begin{aligned}\mathbf{G}_1^l &\rightharpoonup \mathbf{0} && \text{weakly in } L^{r'}(Q_0)^{d\times d},\\ \mathbf{G}_2^l &\to \mathbf{0} && \text{strongly in } L^q(Q_0)^{d\times d},\\ \boldsymbol{u}^l &\to \boldsymbol{u} && \text{strongly in } L^q(Q_0)^d.\end{aligned}$$

Consequently, the assumptions of Lemma 3.1.1 are satisfied. It now suffices to prove for an arbitrary $\theta \in (0,1)$ that

$$\limsup_{l\to\infty} \int_{\frac{1}{8}Q_0} [(\mathbf{D}(\boldsymbol{u}^l) - \boldsymbol{\mathcal{D}}(\cdot,\mathbf{S})) : (\mathbf{S}^l - \mathbf{S})]^\theta \leq 0, \tag{3.34}$$

Once this has been shown, Chacon's biting lemma and Vitali's convergence theorem will imply, together with Lemma 2.4.1, that $(\mathbf{D},\mathbf{S}) \in \mathcal{A}(\cdot)$ almost everywhere in $\frac{1}{8}Q_0$ (see the details e.g. in [BGMŚ12]). From here then the result follows by observing that Q can be covered by a union of such cylinders (e.g. by using a Whitney covering).

In order to prove (3.34), first let $\mathcal{B}_{\lambda_{l,j}} \subset \Omega$ be the family of open sets and let $\{\boldsymbol{e}^{l,j}\}_{l,j\in\mathbb{N}}$ be the sequence of Lipschitz truncations described in Lemma 3.1.1. If we define

$$H^l(\cdot) := (\mathbf{D}(\boldsymbol{u}^l) - \boldsymbol{\mathcal{D}}(\cdot,\mathbf{S})) : (\mathbf{S}^l - \mathbf{S}) \in L^1(Q), \tag{3.35}$$

then we have by Hölder's inequality that

$$\int_{\frac{1}{8}Q_0} |H^l|^\theta \leq |Q|^{1-\theta} \left(\int_{\frac{1}{8}Q_0 \setminus \mathcal{B}_{\lambda_{l,j}}} H^l \right)^\theta + |\mathcal{B}_{\lambda_{l,j}}|^{1-\theta} \left(\int_{\frac{1}{8}Q_0} H^l \right)^\theta.$$

The second term on the right-hand side can be dealt with easily, since H^l is bounded uniformly in $L^1(Q)$ thanks to the a priori estimate (3.27), and the properties described in Lemma 3.1.1 imply that

$$\limsup_{l\to\infty} |\mathcal{B}_{\lambda_{l,j}}|^{1-\theta} \le \limsup_{l\to\infty} |\lambda_{l,j}^r \mathcal{B}_{\lambda_{l,j}}|^{1-\theta} \le c 2^{-j(1-\theta)}, \quad \text{for } j \ge j_0, \tag{3.36}$$

where c is a positive constant. For the first term, observe that

$$\begin{aligned}
&\int_{\frac{1}{8}Q_0\setminus \mathcal{B}_{\lambda_{l,j}}} H^l = \int_{\frac{1}{8}Q_0} H^l \, \zeta \, \mathbb{1}_{\mathcal{B}^c_{\lambda_{l,j}}} \\
&= \int_{\frac{1}{8}Q_0} \mathbf{D}(\boldsymbol{e}^l) : (\mathbf{S}^l - \mathbf{S}) \, \zeta \, \mathbb{1}_{\mathcal{B}^c_{\lambda_{l,j}}} + \int_{\frac{1}{8}Q_0\setminus \mathcal{B}_{\lambda_{l,j}}} (\mathbf{D}(\boldsymbol{u}) - \boldsymbol{\mathcal{D}}(\cdot, \mathbf{S})) : (\mathbf{S}^l - \mathbf{S}) \\
&\le \left| \int_{\frac{1}{8}Q_0} \mathbf{D}(\boldsymbol{e}^{l,j}) : \mathbf{G}_1^l \, \zeta \, \mathbb{1}_{\mathcal{B}^c_{\lambda_{l,j}}} \right| + \left| \int_{\frac{1}{8}Q_0} (\mathbf{D}(\boldsymbol{u}) - \boldsymbol{\mathcal{D}}(\cdot, \mathbf{S})) : (\mathbf{S}^l - \mathbf{S}) \right| \\
&+ \left| \int_{\mathcal{B}_{\lambda_{l,j}}} (\mathbf{D}(\boldsymbol{u}) - \boldsymbol{\mathcal{D}}(\cdot, \mathbf{S})) : (\mathbf{S}^l - \mathbf{S}) \right|,
\end{aligned}$$

where $\zeta \in C^\infty_{0,\mathrm{div}}(\frac{1}{6}Q_0)$ is the function introduced in Lemma 3.1.1. Taking $\limsup_{l\to\infty}$ the assertion follows by taking $j \to \infty$. In particular, we used for the first term Lemma 3.1.1 part 6, with $\boldsymbol{H} = \mathbf{0}$, for the second term the weak convergence of $\mathbf{S}^l$ and for the third term the fact that $\{\mathbf{S}^l\}_{l\in\mathbb{N}}$ is bounded, together with (3.36). To conclude the proof, note that the fact that $\boldsymbol{u}$ is divergence-free and Assumption (A6) imply that $\mathrm{tr}(\mathbf{S}) = 0$, and so $\mathbf{S} \in L^{r'}_{\mathrm{sym,tr}}(\Omega)^{d\times d}$. □

Remark 3.1.6. *The same approach used in Theorem 3.1.5 can be used to define a 3-field formulation for the steady problem and the unsteady problem without convection and the proof remains valid with some simplifications; for instance, for the steady system without the convective term, only the indices k and n are needed. Furthermore, in those cases the convergence of the sequence of discrete pressures can be guaranteed in the corresponding Lebesgue spaces.*

Remark 3.1.7. *The argument used to prove the existence of the discrete solutions is more involved here than in the works [DKS13, BGMŚ09], because the coercivity with respect to $\|\boldsymbol{u}_j^{k,n,m,l}\|_{W^{1,r}(\Omega)}$ cannot be deduced from Formulation $\check{\mathrm{A}}_{k,n,m,l}$ by simply testing with the solution. An alternative approach could be to include in the equation an additional diffusion term of the form:*

$$\frac{1}{k}\int_\Omega |\mathbf{D}(\boldsymbol{u}_j^{k,n,m,l})|^{r-2}\mathbf{D}(\boldsymbol{u}_j^{k,n,m,l}) : \mathbf{D}(\boldsymbol{v}),$$

which would be completely acceptable if we only cared about the existence of weak solutions, but is undesirable from the point of view of the computation of the finite element approximations, since it introduces an additional nonlinearity in the discrete problem.

Remark 3.1.8. *In the proof of Theorem 3.1.5 the limits $k \to \infty$, $(n, m) \to \infty$ and $l \to \infty$ were taken successively. In contrast to the steady case considered in [DKS13], here it is not known whether we can take the limits at once. The result is likely to hold as well, but the proof would require a discrete version of the parabolic Lipschitz truncation, which is not available at the moment.*

Remark 3.1.9. *In case the symmetric velocity gradient is a quantity of interest, the approach presented here can be easily extended to a four-field formulation with unknowns $(\mathbf{D}, \mathbf{S}, \boldsymbol{u}, p)$. The only additional assumption needed in that case would be an inf-sup condition of the form:*

$$\inf_{\boldsymbol{\sigma} \in \Sigma^n} \sup_{\boldsymbol{\tau} \in \Sigma^n} \frac{\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}}{\|\boldsymbol{\sigma}\|_{L^{s'}(\Omega)} \|\boldsymbol{\tau}\|_{L^s(\Omega)}} \geq \delta_s, \tag{3.37}$$

where $\delta_s > 0$ is independent of n.

3.2 Numerical experiments

According to the analysis carried out in the previous section, the addition of the penalty term is necessary when $r \in (\frac{2d}{d+2}, \frac{3d+2}{d+2})$. However, in the examples we observed that the method converges regardless of whether the penalty term is present or not (a comparison is carried out in Section 3.2.2). This could be an indication that the requirement to include this penalty term is only a technical obstruction and that there might be a different approach to showing convergence of the numerical method that could avoid its inclusion in the numerical method. On the other hand, it could also be the case that exact solutions with more severe singularities than the ones considered in our numerical experiments are needed to demonstrate pathological behaviour. In any case, it appears that in most applications the penalty term can be safely omitted and for this reason it is not discussed in the numerical examples below (excepting Section 3.2.2).

3.2.1 Power-law fluid and orders of convergence

The framework presented in this work is so broad that in general it is not possible to guarantee uniqueness of solutions; in particular it is not clear how error estimates could be obtained. However, as this computational example will show, the discrete formulations presented here appear to recover the expected orders of convergence in the cases where these orders are known.

In the first part of this numerical experiment we solved the steady problem without convection with a regularised power-law constitutive relation (as stated in Remark 3.1.6, the same 3-field approximation can be applied in this setting):

$$\mathsf{S} = \boldsymbol{\mathcal{S}}(\mathsf{D}) := 2\nu\left(\varepsilon^2 + |\mathsf{D}^2|\right)^{\frac{r-2}{2}}\mathsf{D}, \tag{3.38}$$

where $r > 1$ and $\varepsilon, \nu > 0$. This is one of the most common non-Newtonian models that present a power-law structure (note that for $r = 2$ we recover the Newtonian model), and has the advantage that it is not singular at the origin (i.e. when $\mathsf{D} = \mathbf{0}$), unlike the usual power-law constitutive relation. Observe that the constitutive relation is smooth, and therefore only the limit $n \to \infty$ is needed in the results from the previous section. The problem was solved on the unit square $\Omega = (0,1)^2$ with a Dirichlet boundary condition for the velocity defined so as to match the value of the exact solution, which was chosen as:

$$\boldsymbol{u}(\boldsymbol{x}) = |\boldsymbol{x}|^{a-1}(x_2, -x_1)^\top, \qquad p(\boldsymbol{x}) = |\boldsymbol{x}|^b, \tag{3.39}$$

where a, b are parameters used to control the smoothness of the solutions. Define the auxiliary function $\boldsymbol{F} := \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\text{sym}}$ as:

$$\boldsymbol{F}(\mathsf{B}) := (\varepsilon + |\mathsf{B}^{\text{sym}}|)^{\frac{r-2}{2}}\mathsf{B}^{\text{sym}}, \tag{3.40}$$

where $\mathsf{B}^{\text{sym}} := \frac{1}{2}(\mathsf{B} + \mathsf{B}^T)$. In [BBDR12, Hir13] it was proved for systems of the form (3.38) that if $\boldsymbol{F}(\mathsf{D}(\boldsymbol{u})) \in W^{1,2}(\Omega)^{d\times d}$ and $p \in W^{1,r'}(\Omega)$ then the following error estimates hold:

$$\|\boldsymbol{F}(\mathsf{D}(\boldsymbol{u})) - \boldsymbol{F}(\mathsf{D}(\boldsymbol{u}^n))\|_{L^2(\Omega)} \leq c h_n^{\min\{1, \frac{r'}{2}\}},$$
$$\|p - p^n\|_{L^{r'}(\Omega)} \leq c h_n^{\min\{\frac{2}{r'}, \frac{r'}{2}\}}.$$

In our case, the conditions $\boldsymbol{F}(\mathsf{D}(\boldsymbol{u})) \in W^{1,2}(\Omega)^{d\times d}$ and $p \in W^{1,r'}(\Omega)$ amount to requiring that $a > 1$ and $b > \frac{2}{r} - 1$. These parameters were then chosen to be $a = 1.01$ and $b = \frac{2}{r} - 0.99$ in order to be close to the regularity threshold. We discretised this problem with the Scott–Vogelius element for the velocity and pressure and discontinuous piecewise polynomials for the stress variables:

$$\begin{aligned}
\Sigma^n &= \{\boldsymbol{\sigma} \in L^\infty_{\mathrm{sym,tr}}(\Omega)^{d\times d} \,:\, \boldsymbol{\sigma}|_K \in \mathbb{P}_k(K)^{d\times d}, \text{ for all } K \in \mathcal{T}_n\},\\
V^n &= \{\boldsymbol{w} \in W^{1,r}(\Omega)^d \,:\, \boldsymbol{w}|_{\partial\Omega} = \boldsymbol{u},\, \boldsymbol{w}|_K \in \mathbb{P}_{k+1}(K)^d \text{ for all } K \in \mathcal{T}_n\},\\
M^n &= \{q \in L^\infty(\Omega) \,:\, q|_K \in \mathbb{P}_k(K) \text{ for all } K \in \mathcal{T}_n\}.
\end{aligned}$$

The problem was solved using `firedrake` [RHM+16] with $\nu = 0.5$, $\varepsilon = 10^{-5}$ and $k = 1$ on a barycentrically refined mesh (obtained using `gmsh` [GR09]) to guarantee inf-sup stability. The discretised nonlinear problems were linearised using Newton's method with the L^2 line search algorithm of PETSc [BAA+17, BKST15]; the Newton solver was deemed to have converged when the Euclidean norm of the residual fell below 1×10^{-8}. The linear systems were solved with a sparse direct solver from the `umfpack` library [Dav04]. In the implementation, the uniqueness of the pressure was recovered not by using a zero mean condition but rather by orthogonalising against the nullspace of constants. The experimental orders of convergence in the different norms are shown in Tables 3.1 and 3.2 (note that the tables do not contain the values of the numerical error, but rather the order of convergence corresponding to the norm indicated in each column).

h_n	$\|\boldsymbol{F}(\mathbf{D}(\boldsymbol{u}))\|_{L^2(\Omega)}$	$\|\boldsymbol{u}\|_{W^{1,r}(\Omega)}$	$\|p\|_{L^{r'}(\Omega)}$	$\|\mathbf{S}\|_{L^{r'}(\Omega)}$
0.5	0.9075	1.0180	0.3647	0.6692
0.25	0.9803	1.2160	0.5396	0.6697
0.125	1.0023	1.2975	0.6565	0.6713
0.0625	1.0062	1.3205	0.6706	0.6716
0.03125	1.0071	1.3319	0.6715	0.6716
Expected	1.0	-	0.667	-

Table 3.1: Experimental order of convergence for the steady problem without convection with $r = 1.5$.

From Tables 3.1 and 3.2 it can be seen that the algorithm recovers the expected orders of convergence. In the case of the stress we obtain the same order as for the pressure, which seems natural from the point of view of the equation. In [Hir13] it is claimed that for $r < 2$ the order of convergence for the velocity should be equal to 1; in our numerical simulations the experimental order of convergence seems to approach $\frac{2}{r}$, which is slightly larger than 1. This difference may be due to the fact

h_n	$\lVert \boldsymbol{F}(\mathbf{D}(\boldsymbol{u}))\rVert_{L^2(\Omega)}$	$\lVert \boldsymbol{u}\rVert_{W^{1,r}(\Omega)}$	$\lVert p\rVert_{L^{r'}(\Omega)}$	$\lVert \mathbf{S}\rVert_{L^{r'}(\Omega)}$
0.5	0.9132	0.9361	0.4955	0.8434
0.25	0.9826	1.0652	0.7271	0.8822
0.125	1.0040	1.1073	0.8671	0.8948
0.0625	1.0078	1.1167	0.8916	0.8966
0.03125	1.0087	1.1197	0.8959	0.8968
Expected	1.0	-	0.889	-

Table 3.2: Experimental order of convergence for the steady problem without convection with $r = 1.8$.

that in [Hir13] the author works with piecewise linear elements for the velocity while here quadratic elements were employed.

In the second part of the experiment we employed again the power-law constitutive relation (3.38), but now considering the unsteady system (3.1). The right-hand side, initial condition and boundary condition were chosen so as to match the ones defined by the exact solution:

$$\boldsymbol{u}(t,\boldsymbol{x}) = t|\boldsymbol{x}|^{a-1}(x_2,-x_1)^\top, \quad p(t,\boldsymbol{x}) = t^2|\boldsymbol{x}|^b.$$

In [ER18], the following error estimate for the approximation of time-dependent systems of this form, but without convection, was obtained for $r \in [\frac{2d}{d+2},\infty)$:

$$\lVert \boldsymbol{u} - \overline{\boldsymbol{u}}^{n,m}\rVert_{L^\infty(0,T;L^2(\Omega))} + \lVert \boldsymbol{F}(\mathbf{D}(\boldsymbol{u})) - \boldsymbol{F}(\mathbf{D}(\overline{\boldsymbol{u}}^{n,m}))\rVert_{L^2(Q)} \leq c\left(\tau_m + h_n^{\min\{1,\frac{2}{r}\}}\right),$$

assuming that $\boldsymbol{u}_0 \in W^{1,r}_{0,\mathrm{div}}(\Omega)^d$ and that the following additional regularity properties of the solution and the data hold:

$$\begin{gathered}\lVert \nabla \boldsymbol{F}(\mathbf{D}(\boldsymbol{u}_0))\rVert_{L^2(\Omega)} + \lVert \nabla \mathbf{S}(\mathbf{D}(\boldsymbol{u}_0))\rVert_{L^2(\Omega)} \leq c,\\ \lVert \boldsymbol{u}\rVert_{W^{1,2}(0,T;L^2(\Omega))} + \lVert \boldsymbol{u}\rVert_{L^2(0,T;W^{2,2}(\Omega))} + \lVert \boldsymbol{F}(\mathbf{D}(\boldsymbol{u}))\rVert_{L^2(0,T;W^{1,2}(\Omega))} \leq c.\end{gathered}$$

The same order of convergence was obtained in [BDR15] for $r \in (\frac{3}{2},2]$ in 3D for a semi-implicit discretisation of the unsteady system with convection assuming that $\boldsymbol{u}_0 \in W^{2,2}_{0,\mathrm{div}}(\Omega)^d$, $\operatorname{div}\mathbf{S}(\mathbf{D}(\boldsymbol{u}_0)) \in L^2(\Omega)^d$ and that the slightly different regularity assumptions hold:

$$\lVert \partial_t \boldsymbol{u}\rVert_{L^\infty(0,T;L^2(\Omega))} + \lVert \boldsymbol{F}(\mathbf{D}(\boldsymbol{u}))\rVert_{W^{1,2}(Q)} + \lVert \boldsymbol{F}(\mathbf{D}(\boldsymbol{u}))\rVert_{L^{2((5r-6)/(2-r))}(0,T;W^{1,2}(\Omega))} \leq c.$$

The problem was solved until the final time $T = 0.1$ with the same parameters as above; observe that this choice of parameters guarantees that the required regularity

h_n	τ_m	$\|\boldsymbol{F}(\mathbf{D}(\boldsymbol{u}))\|_{L^2(Q)}$	$\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))}$
0.5	0.001	0.9226	1.8703
0.25	0.0005	0.9865	1.9564
0.125	0.00025	1.0057	1.9497
0.0625	0.000125	1.0084	1.9440
0.03125	0.0000625	1.0075	1.9451
Expected		1.0	1.0

Table 3.3: Experimental order of convergence for the unsteady problem with $r = 1.7$.

properties are satisfied. Table 3.3 shows the experimental order of convergence for $r = 1.7$. The order of convergence for the natural norm $\|\boldsymbol{F}(\mathbf{D}(\boldsymbol{u}))\|_{L^2(Q)}$ agrees with the one expected from the theoretical results, while for the velocity we obtain a higher order. This is again likely to be due to the fact that quadratic elements were employed for the velocity variable, while the analysis was performed for linear elements.

3.2.2 Role of the penalty term

In this computational experiment we investigate the role of the penalty term in the algorithm, to explore whether its presence is essential to ensure convergence. Similarly to Section 3.2.1 we consider the steady problem first. The same exact solution was employed, because it allows us to carefully select its regularity. In this case Taylor–Hood elements were employed for the velocity and pressure, and discontinuous piecewise polynomials for the stress:

$$
\begin{aligned}
\Sigma^n &= \{\boldsymbol{\sigma} \in L^\infty_{\mathrm{sym}}(\Omega)^{d\times d} \,:\, \boldsymbol{\sigma}|_K \in \mathbb{P}_1(K)^{d\times d}, \text{ for all } K \in \mathcal{T}_n\},\\
V^n &= \{\boldsymbol{w} \in W^{1,r}(\Omega)^d \,:\, \boldsymbol{w}_\tau|_{\partial\Omega_1} = 0,\, \boldsymbol{w}|_{\partial\Omega_2} = \mathbf{0},\, \boldsymbol{w}|_K \in \mathbb{P}_2(K)^d \text{ for all } K \in \mathcal{T}_n\},\\
M^n &= \{q \in L^\infty(\Omega) \cap C(\overline{\Omega}) \,:\, q|_k \in \mathbb{P}_1(K) \text{ for all } K \in \mathcal{T}_n\}.
\end{aligned}
$$

This question is only relevant when the discretely divergence-free elements are not pointwise divergence-free, because otherwise the condition $r > \frac{2d}{d+2}$ is sufficient to allow us to pass to the limit in the convective term. In the steady case, without the penalty term and with elements that are not exactly divergence-free, the convergence of the finite element approximations still holds (see Remark 3.1.6), but assuming the stronger assumption $r > \frac{2d}{d+1}$ (cf. [DKS13]). According to this, the addition of the penalty term is then necessary in the convergence analysis when the elements are not exactly divergence-free and $r \in (\frac{2d}{d+2}, \frac{2d}{d+1}]$. Table 3.4 shows the experimental orders of convergence for $r = 1.3$ (just like in Section 3.2.1, the table shows not the numerical error, but the experimental order of convergence).

In the experiment for the time-dependent problem we chose in this case the steady state solution (3.39) with the same parameters described above and used it to define the initial and boundary conditions. In this case, our convergence analysis dictates that the addition of the penalty term is necessary when $r \in (\frac{2d}{d+2}, \frac{3d+2}{d+2})$; however, the result is expected to hold for $r \in (\frac{2(d+1)}{d+2}, \frac{3d+2}{d+2})$ as well (see Remark 3.1.8). We therefore chose a value of r in the interval $(\frac{2d}{d+2}, \frac{2(d+1)}{d+2}]$. The experimental orders of convergence for this case are shown in Table 3.5.

h_n	$\\|\boldsymbol{F}(\mathsf{D}(\boldsymbol{u}))\\|_{L^2(\Omega)}$	$\\|p\\|_{L^{r'}(\Omega)}$
0.5	0.97295673154	1.91148217955
0.25	1.00506435728	0.470815332994
0.125	1.0089872966	0.51434542432
0.0625	1.00879694502	0.472841717098
0.03125	1.00895395592	0.463776304819
Expected	1.0	0.461

Table 3.4: Experimental order of convergence for the steady problem with $r = 1.3$.

h_n	τ_m	With Penalty Term	Without Penalty Term
0.5	0.005	9.73599147231	5.502165863559
0.25	0.0025	1.008703378392	0.98183996942
0.125	0.00125	1.00651090357	1.00190875446
0.0625	0.000625	1.0154632500	1.00811647604
0.03125	0.0003125	1.028436230	1.01326314547
Expected		1.0	1.0

Table 3.5: Experimental order of convergence for the $\\|\boldsymbol{F}(\mathsf{D}(\boldsymbol{u}))\\|_{L^2(Q)}$ norm for the full problem with $r = 1.3$.

What we see in these experiments is that the method converges regardless of whether there is a penalty term or not. As mentioned at the beginning of this section, this could be an indication that the requirement to include this penalty term is only a technical obstruction and that there might be a different approach to showing convergence of the numerical method that could avoid it. In any case, we believe that in most applications the penalty term can be safely omitted.

3.2.3 Navier–Stokes/Euler activated fluid

In this section we will consider the classical lid–driven cavity problem with the non–standard constitutive relation:

$$\begin{cases} \begin{cases} \mathsf{D} = \delta_s \frac{\mathsf{S}}{|\mathsf{S}|} + \frac{1}{2\nu}\mathsf{S}, & \text{if } |\mathsf{D}| \geq \delta_s, \\ \mathsf{S} = 0, & \text{if } |\mathsf{D}| < \delta_s, \end{cases} & \text{if } (x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 \leq (\frac{3}{8})^2, \\ \qquad \mathsf{D} = \frac{1}{2\nu}\mathsf{S}, & \qquad \text{otherwise,} \end{cases} \tag{3.41}$$

where $\nu > 0$ is the viscosity and $\delta_s \geq 0$. Constitutive relations of this type were analysed for the first time in [BMR20]; the relation (3.41) is an example of an activated fluid that in the middle of the domain transitions between a Newtonian fluid (i.e. Navier–Stokes) and an inviscid fluid (i.e. Euler) depending on the magnitude of the symmetric velocity gradient. As mentioned in Chapter 1, the fact that we can swap the roles of the stress and the symmetric velocity gradient in constitutive relations without any problem is a significant advantage of the framework presented here.

The problem was solved on the unit square $\Omega = (0,1)^2$ with the rest state as the initial condition and with the following boundary conditions:

$$\begin{aligned} &\partial\Omega_1 = (0,1)\times\{1\}, && \partial\Omega_2 := \partial\Omega \setminus \partial\Omega_1, \\ &\boldsymbol{u} = \boldsymbol{0} && \text{on } (0,T)\times\partial\Omega_2, \\ &\boldsymbol{u} = (x^2(1-x)^2 16y^2, 0)^\top && \text{on } (0,T)\times\partial\Omega_1. \end{aligned}$$

Although (3.41) has a complicated form, there is a continuous (in D) selection available:

$$\mathsf{S} = \boldsymbol{\mathcal{S}}(x,y,\mathsf{D}) := \begin{cases} 2\nu\left(|\mathsf{D}| - \delta_s \mathbb{1}_{B_{3/8}(1/2)}(x,y)\right)^+ \frac{\mathsf{D}}{|\mathsf{D}|}, & \text{if } |\mathsf{D}| \neq 0, \\ \boldsymbol{0}, & \text{if } |\mathsf{D}| = 0. \end{cases} \tag{3.42}$$

While the selection stated in (3.42) is already continuous in D, Newton's method requires Fréchet-differentiability of $\boldsymbol{\mathcal{S}}$ with respect to D and the constitutive law is not smooth when $|(x-\frac{1}{2}, y-\frac{1}{2})| < \frac{3}{8}$; therefore some regularisation was required for the purpose of applying Newton's method (an alternative would have been to use a non-smooth generalisation such as a semismooth Newton method). For this problem we chose a Papanastasiou-like regularisation (cf. [Pap87]); the Papanastasiou regularisation has been successfully applied to several problems with Bingham rheology [CGA+05, DT03, MH04]. The regularised constitutive relation reads:

$$\mathsf{D} = \left(\frac{\delta_s(1-\exp(-M|\mathsf{S}|))}{|\mathsf{S}|} + \frac{1}{2\nu}\right)\mathsf{S} \quad \text{for } (x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2 \leq (\tfrac{3}{8})^2, \tag{3.43}$$

where $M > 0$ is the regularisation parameter (as $M \to \infty$ we recover the constitutive relation (3.41), see Figure 3.1); note that this is not related to the regularisation (2.15), which has the goal of turning the *measurable* selection into a continuous function. For the velocity and pressure we used Scott–Vogelius elements and discontinuous piecewise polynomials were used for the stress (cf. 3.2.1); the problem was implemented in `firedrake` with $k = 1$, $\nu = \frac{1}{2}$, using the same parameters for the linear and nonlinear solvers described in the previous section, and continuation was employed to reach the values $M = 200$ and $\delta_s = 2.5$; more precisely, the problem was initially solved with $M = 100$ and $\delta_s = 0$ and that solution was used as the Newton guess for the problem with $M + 1$ and $\delta_s + 0.05$, repeating the procedure until the desired values were reached. The time step was chosen as $\tau_m = 5 \times 10^{-6}$ and the algorithm was applied until the L^2 norm of the difference of solutions at subsequent time steps was less than 1×10^{-6}.

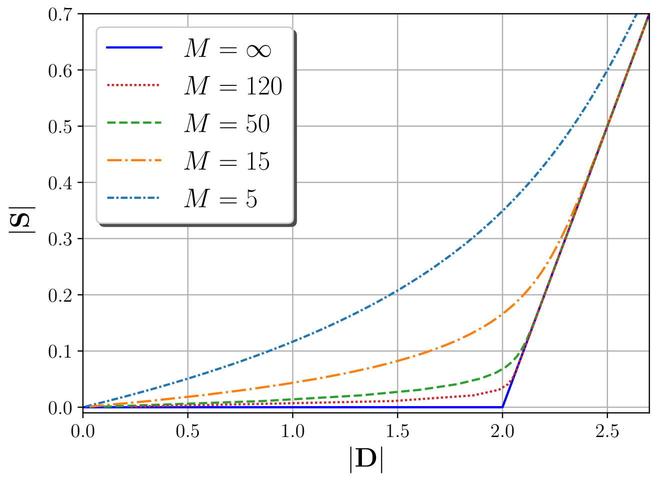


Figure 3.1: Regularised constitutive relation for different values of M and $\delta_s = 2$.

Note that when the 'yield strain' parameter δ_s vanishes, we recover the usual Navier–Stokes system. On the other end, if δ_s is taken to be very large this could be taken as an approximation of the incompressible Euler system in the center of the square; notice how in Figure 3.2 the fluid picks up more speed in the middle of the domain when $\delta_s > 0$ due to the absence of viscosity. This could be an attractive approach to simulating the effects of boundary layers, because it is backed up by a rigorous convergence result; near the boundary the fluid could behave in a Newtonian way and far away δ_s could be taken arbitrarily large so as to make the effects of the viscosity negligible. This is just one of the possibilities that are yet to be explored within this framework of implicitly constituted fluids and mixed formulations and will be studied in more depth in future work.

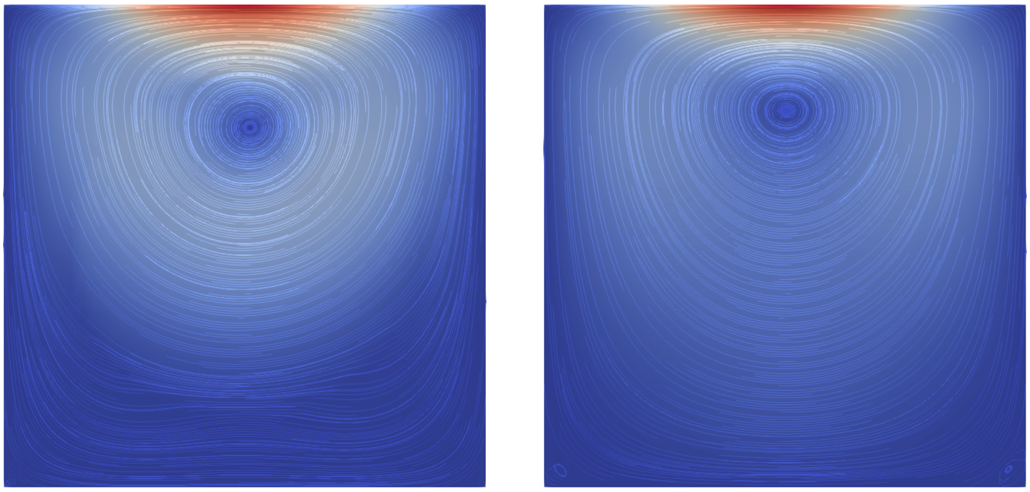


Figure 3.2: Streamlines of the steady state for the problem with $\delta_s = 2.5$ (left) and the Newtonian problem (right).

Figure 3.3 shows the magnitudes of $\mathbf{S}$ and $\mathbf{D}$ along the line $x = 0.65$ for the steady state of the non-Newtonian problem; it can be clearly seen that the stress is negligibly small for low values of the symmetric velocity gradient in the center of the square and it then suddenly becomes proportional to it. This transition is not the sharpest in the figure because the regularisation parameter M was not taken sufficiently large, but in the limit this would recover the non-smooth relation. In a sense this is similar to solving a Navier–Stokes problem with high Reynolds number, so for high values of M some stabilisation would be required in order to solve this system efficiently (even more so if the Newtonian fluid outside of the activation region also has a high Reynolds number); this will be the subject of future research.

3.2.4 Cessation of the Couette flow of a Bingham fluid

The flow between two parallel plates induced by the movement at constant speed of one of the plates receives the name of (plane) Couette flow. It is one of the few examples of a configuration that allows us to find an exact solution for the steady Navier–Stokes equations and it is well known that this solution has a linear profile. In this numerical experiment we will take the Couette flow as the initial condition and investigate the behaviour of the system when the plates stop moving. Physically it is expected that the viscosity and no-slip boundary condition will slow down the

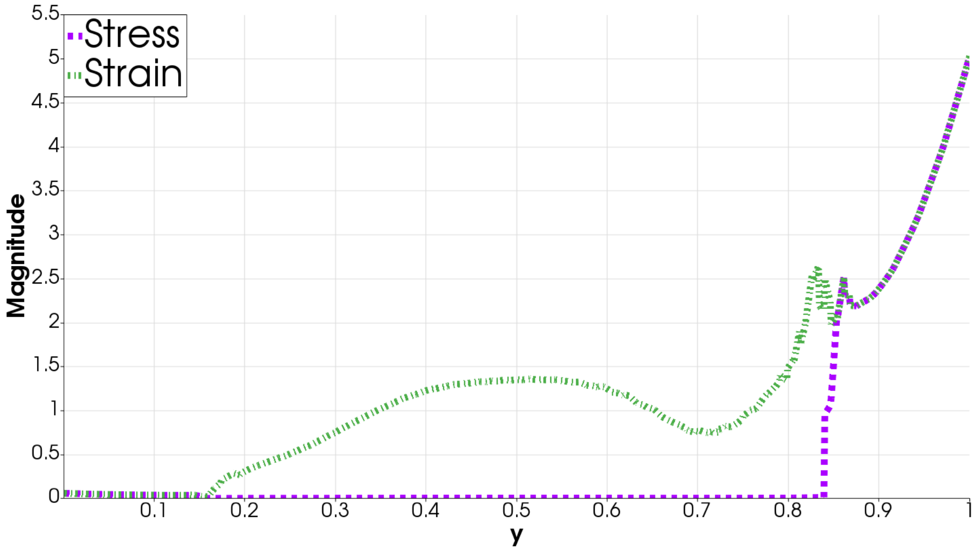


Figure 3.3: Magnitude of $\mathbf{S}$ and $\mathbf{D}$ at $x = 0.65$ for the problem with $\delta_s = 2.5$.

flow until it finally stops; it can be seen in [PGA99] that in the Newtonian case the flow does reach the rest state, albeit in infinite time.

In this section we will solve system (3.1) with the Bingham constitutive relation (c.f. (1.12)):

$$\begin{cases} \mathbf{S} = \tau_y \frac{\mathbf{D}}{|\mathbf{D}|} + 2\nu \mathbf{D}, & \text{if } |\mathbf{S}| \geq \tau_y, \\ \mathbf{D} = 0, & \text{if } |\mathbf{S}| < \tau_y, \end{cases}$$

where $\nu > 0$ is the viscosity and $\tau_y \geq 0$ is the yield stress. Interestingly, viscoplastic fluids in the configuration described above reach the rest state in a finite time and there are theoretical upper bounds for the so called *cessation time* (see [Glo84, HMP02]), which makes this a good problem to test the numerical algorithm. Just as in the previous section, for this problem there is also a continuous selection available:

$$\mathbf{D} = \boldsymbol{\mathcal{D}}(\mathbf{S}) := \begin{cases} \frac{1}{2\nu}(|\mathbf{S}| - \tau_y)^+ \frac{\mathbf{S}}{|\mathbf{S}|}, & \text{if } |\mathbf{S}| \neq 0, \\ \mathbf{0}, & \text{if } |\mathbf{S}| = 0. \end{cases} \tag{3.44}$$

For this experiment we again applied the Papanastasiou regularisation to the non-smooth constitutive relation, in order to be able to apply Newton's method. After nondimensionalisation this regularised constitutive law takes the form (compare with (3.43)):

$$\mathbf{S} = \left(\frac{\mathrm{Bn}}{|\mathbf{D}|}(1 - \exp(-M|\mathbf{D}|)) + 1 \right) \mathbf{D}, \tag{3.45}$$

where $\mathrm{Bn} = \frac{\tau_y L}{\nu U}$ is the Bingham number (here U and L are a characteristic velocity and length of the problem, respectively), and $M > 0$ is the regularisation parameter (as $M \to \infty$ we recover the non–smooth relation; compare with 3.1). The problem was solved on the unit square $\Omega = (0,1)^2$ with the following boundary conditions:

$$\partial\Omega_1 = \{0\} \times (0,1) \cup \{1\} \times (0,1), \qquad \partial\Omega_2 := (0,1) \times \{1\} \cup (0,1) \times \{0\},$$
$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } (0,T) \times \partial\Omega_2,$$
$$\boldsymbol{u}_\tau = 0 \qquad \text{on } (0,T) \times \partial\Omega_1,$$
$$-p + \mathbf{S}\boldsymbol{n} \cdot \boldsymbol{n} = 0, \qquad \text{on } (0,T) \times \partial\Omega_1,$$

where $\boldsymbol{u}_\tau$ denotes the component of the velocity tangent to the boundary and $\boldsymbol{n}$ is the unit vector normal to the boundary. The initial condition was taken as a standard Couette flow:

$$\boldsymbol{u}(0, \boldsymbol{x}) = (1 - x_2, 0)^\top.$$

For the velocity and pressure we used Taylor–Hood elements and discontinuous piecewise polynomials for the stress (cf. Section 3.2.2). This problem was implemented in FEniCS [LMW11] using the same parameters for the nonlinear and linear solvers described in the previous section, with $k = 1$ and a timestep τ_m between 5×10^{-7} and 1×10^{-6} for the different values of the Bingham number. We quantify the change in the flow through the volumetric flow rate (observe that it is constant in x_1):

$$Q(t) := \int_0^1 (1,0) \cdot \boldsymbol{u}(t, \boldsymbol{x}) \,\mathrm{d}x_2,$$

whose evolution in time is shown in Figure 3.4 for different values of the Bingham number. An exponential decay of the flow rate is observed in Figure 3.4, while for positive values of the Bingham number this decay is much faster; these results agree with the ones reported in [HMP02, CGA$^+$05]. In [CGA$^+$05] the problem was solved by integrating a one-dimensional equation for u_2; the framework presented here recovers the results obtained there but at the same time has the advantage that it can be applied to a much broader class of problems and geometries.

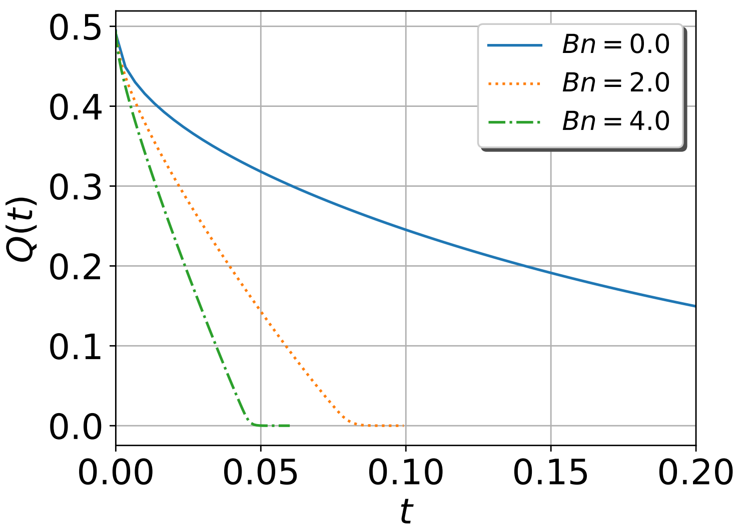


Figure 3.4: Evolution of the volumetric flow rate.

Chapter 4

Augmented Lagrangian Preconditioner: Isothermal Case

In this chapter we will construct a preconditioner for the Newton linearisation of a steady system describing an incompressible fluid:

$$
\begin{aligned}
-\operatorname{div}\mathbf{S} + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p &= \boldsymbol{f} && \text{on } \Omega,\\
\operatorname{div}\boldsymbol{u} &= 0 && \text{on } \Omega,\\
\boldsymbol{u} &= \boldsymbol{u}_b && \text{on } \partial\Omega,
\end{aligned} \tag{4.1a}
$$

where $\boldsymbol{f} \in L^{r'}(\Omega)^d$ and $\boldsymbol{u}_b \in W^{1/r',r}(\partial\Omega)^d$ are given, for some $r > 1$. In order to ensure the uniqueness of the pressure we impose a zero mean constraint $\int_\Omega p = 0$. The system is closed with an implicit constitutive relation of the form

$$
\mathbf{G}(\cdot,\mathbf{S},\mathbf{D}) := \alpha(\cdot,|\mathbf{S}|^2,|\mathbf{D}|^2)\mathbf{D} - \beta(\cdot,|\mathbf{S}|^2,|\mathbf{D}|^2)\mathbf{S} = \mathbf{0}, \tag{4.1b}
$$

where $\alpha,\beta : \Omega\times[0,\infty)^2 \to \mathbb{R}^+$ are positive functions, and $\mathbf{G} : \Omega\times\mathbb{R}^{d\times d}_{\text{sym}}\times\mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^{d\times d}_{\text{sym}}$ satisfies the following assumptions:

(B1) The mapping $(\mathbf{D},\mathbf{S}) \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}} \mapsto \mathbf{G}(x,\mathbf{S},\mathbf{D})$ is Fréchet-differentiable for almost every $x \in \Omega$.

(B2) The mapping $x \in \Omega \mapsto \mathbf{G}(x,\mathbf{S},\mathbf{D})$ belongs to $L^\infty(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})$ for every $(\mathbf{D},\mathbf{S}) \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}}$.

The differentiability assumption (B1) is needed because Newton's method will be applied to linearise the system. Let us denote by $\mathcal{A} : \Omega \to \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}}$ the graph defined by $\mathbf{G}$ (recall (2.12)). We will assume in addition that, for some $r > 1$, $\mathcal{A}(\cdot)$ satisfies assumption (A4) and a stricter version of (A3) from Section 2.4:

(B3) [$\mathcal{A}$ *is a strictly monotone graph*] For every $(\mathbf{D}_1, \mathbf{S}_1), (\mathbf{D}_2, \mathbf{S}_2) \in \mathcal{A}(x)$,

$$(\mathbf{S}_1 - \mathbf{S}_2) : (\mathbf{D}_1 - \mathbf{D}_2) \geq 0,$$

with strict inequality when $(\mathbf{D}_1, \mathbf{S}_1) \neq (\mathbf{D}_2, \mathbf{S}_2)$.

Clearly these statements should hold now almost everywhere in Ω, as opposed to Q (as this is now a problem not involving time). In the weak formulation of this system we look for $(\mathbf{S}, \boldsymbol{u}, p) \in L^{r'}_{\mathrm{sym,tr}}(\Omega)^{d\times d} \times \big(\boldsymbol{u}_0 + W^{1,r}_{0,\mathrm{div}}(\Omega)^d\big) \times L^{\tilde{r}}_0(\Omega)$, where $\tilde{r} := \min\{r', r^*/2\}$, such that

$$\int_\Omega \mathbf{S} : \mathbf{D}(\boldsymbol{v}) - \int_\Omega \boldsymbol{u} \otimes \boldsymbol{u} : \mathbf{D}(\boldsymbol{v}) - \int_\Omega p \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in W^{1,\tilde{r}'}_0(\Omega)^d, \tag{4.2a}$$

$$(\mathbf{D}(\boldsymbol{u}), \mathbf{S}) \in \mathcal{A}(\cdot) \qquad \text{a.e. in } \Omega, \tag{4.2b}$$

$$-\int_\Omega q \operatorname{div} \boldsymbol{u} = 0 \qquad \forall\, q \in L^{r'}(\Omega). \tag{4.2c}$$

Observe that for a constitutive relation of the form (4.1b), the assumptions (B1)–(B3) imply that $\mathcal{A}$ is a *maximal monotone r-graph* (i.e. (A1),(A2),(A3) and (A4) are also satisfied), and under these conditions the results in Chapter 3 guarantee the existence of a weak solution. Using the specific form (4.1b) is not strictly necessary, but it has the advantage that it reduces the number of necessary assumptions, and many models can be expressed in this form, in any case. The strict monotonicity assumption (B3) is required to prevent the Jacobian from becoming singular, which is necessary when employing Newton's method.

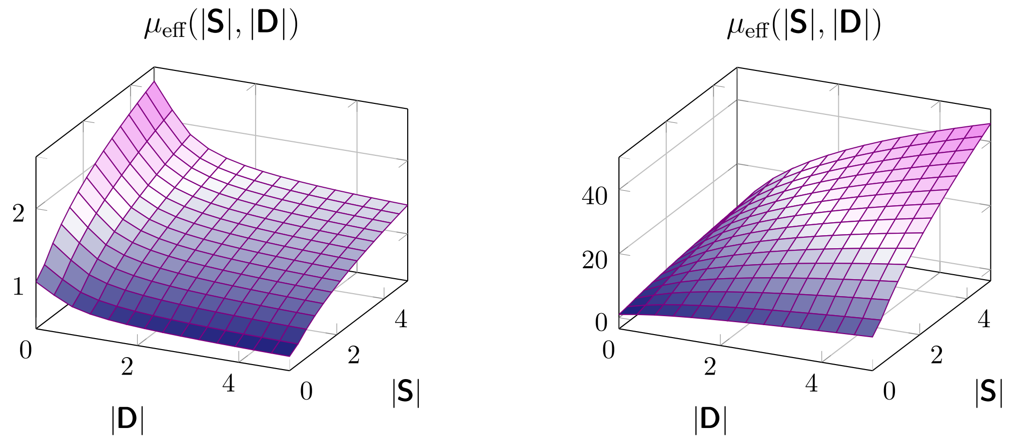
The relation (4.1b) defines a general constitutive law with a power-law structure describing a fluid with an effective viscosity that depends both on $|\mathbf{D}|$ and $|\mathbf{S}|$; in this setting the effective viscosity can be defined as:

$$\mu_{\text{eff}}(\cdot, |\mathbf{S}|, |\mathbf{D}|) := \frac{1}{2} \frac{\alpha(\cdot, |\mathbf{S}|^2, |\mathbf{D}|^2)}{\beta(\cdot, |\mathbf{S}|^2, |\mathbf{D}|^2)}. \tag{4.3}$$

An important example that is captured by the assumptions above is the generalised Carreau–Yasuda constitutive relation:

$$\begin{aligned} \mathbf{G}(\mathbf{S}, \mathbf{D}) = {} & \Big(\beta_1 + (1-\beta_1)(1 + \Gamma_1 |\mathbf{D}|^2)^{\frac{r_1-2}{2}}\Big) \mathbf{D} \\ & - \frac{1}{2\nu} \Big(\beta_2 + (1-\beta_2)(1 + \Gamma_2 |\mathbf{S}|^2)^{\frac{2-r_2}{2(r_2-1)}}\Big) \mathbf{S}, \end{aligned} \tag{4.4}$$

where $r_1, r_2 > 1$, $1 \geq \beta_1, \beta_2 \geq 0$ and $\nu, \Gamma_1, \Gamma_2 > 0$ are given parameters. Note that when $\beta_2 = 1$ the relation (4.4) reduces to the Carreau–Yasuda constitutive relation



(a) $\nu = 1$, $r_1 = 1.2$, $r_2 = 2.5$, $\beta_1 = 0.5$, $\beta_2 = 0$, $\Gamma_1 = \Gamma_2 = 10$.

(b) $\nu = 1$, $r_1 = 2.5$, $r_2 = 3.5$, $\beta_1 = \beta_2 = 0$, $\Gamma_1 = 100$, $\Gamma_2 = 20$.

Figure 4.1: Effective viscosity for the generalised Carreau–Yasuda relation (4.4). Shear-thinning and stress-thickening behaviour can be observed in (a) while (b) presents only thickening behaviour.

(1.8), and when $r_1 = 2 = r_2$ or $\beta_1 = 1 = \beta_2$ it reduces to the usual Newtonian relation $\mathbf{S} = 2\nu\mathbf{D}$. When $r_1 = 2$, the relation (4.4) can for instance capture the models with stress-dependent viscosity of Ellis (1.15) and Glen (1.16). Figure 4.1 shows the behaviour of the effective viscosity for two choices of the parameters.

Another example is given by regularisations of the Bingham constitutive relation (c.f. (1.12)):

$$\begin{cases} \mathbf{S} = \tau_y \frac{\mathbf{D}}{|\mathbf{D}|} + 2\nu\mathbf{D}, & \text{if } |\mathbf{S}| \geq \tau_y, \\ \mathbf{D} = 0, & \text{if } |\mathbf{S}| < \tau_y, \end{cases} \tag{4.5}$$

where $\nu > 0$ and $\tau_y \geq 0$. Note that such a relation can be written using an expression of the form (4.1b); for instance, it could be described using the following functions:

$$\mathbf{G}_1(\mathbf{S}, \mathbf{D}) = 2\nu(\tau_y + |2\nu\mathbf{D}|)\mathbf{D} - |2\nu\mathbf{D}|\mathbf{S}, \tag{4.6a}$$

$$\mathbf{G}_2(\mathbf{S}, \mathbf{D}) = \begin{cases} \mathbf{D} - \frac{1}{2\nu}(|\mathbf{S}| - \tau_y)^+ \frac{\mathbf{S}}{|\mathbf{S}|}, & \text{if } \mathbf{S} \neq \mathbf{0}, \\ \mathbf{D}, & \text{if } \mathbf{S} = 0. \end{cases} \tag{4.6b}$$

However, the expressions in (4.6) do not satisfy the differentiability assumption (B1) and so Newton's method cannot be directly applied. As was done in Section 3.2.4, this difficulty can be addressed by applying a suitable regularisation step. For example, the following functions could be used instead of (4.6):

$$\tilde{\mathbf{G}}_1(\mathbf{S},\mathbf{D}) = 2\nu(\tau_y + |2\nu\mathbf{D}|)\mathbf{D} - \sqrt{|2\nu\mathbf{D}|^2+\varepsilon^2}\mathbf{S}, \tag{4.7a}$$

$$\tilde{\mathbf{G}}_2(\mathbf{S},\mathbf{D}) = \left(2\nu + \frac{\tau_y}{|\mathbf{D}|}\right)(1-\mathrm{e}^{-|\mathbf{D}|/\varepsilon})\mathbf{D} - \mathbf{S}, \tag{4.7b}$$

where ε is a positive small parameter. The relation defined by (4.7b) is the Papanastasiou regularisation (c.f. (3.45)), and while (4.7a) is related to the Bercovier–Engelman regularisation [BE80], it is not usually written in this manner. This illustrates the freedom that the framework presented here offers; it is up to the practitioner to find the most convenient expression for a given constitutive relation.

Let us now take a barycentrically refined triangulation $\mathcal{T}_n$ of Ω. In this chapter we will employ the Scott–Vogelius finite element pair for the velocity and pressure, and discontinuous polynomials for the stress:

$$\begin{aligned}
\Sigma^n &= \{\boldsymbol{\sigma} \in L^\infty_{\mathrm{sym,tr}}(\Omega)^{d\times d} \,:\, \boldsymbol{\sigma}|_K \in \mathbb{P}_k(K)^{d\times d},\ \text{for all } K \in \mathcal{T}_n\},\\
V^n &= \{\boldsymbol{w} \in W^{1,r}(\Omega)^d \,:\, \boldsymbol{w}|_{\partial\Omega} = \boldsymbol{u}_b,\ \boldsymbol{w}|_K \in \mathbb{P}_{k+1}(K)^d \text{ for all } K \in \mathcal{T}_n\},\\
M^n &= \{q \in L^\infty_0(\Omega) \,:\, q|_k \in \mathbb{P}_k(K) \text{ for all } K \in \mathcal{T}_n\},
\end{aligned}$$

where $k \geq d$. The fact that we can work with traceless stresses, and thus fewer degrees of freedom, stems from the fact that the discretely divergence-free velocities with the Scott–Vogelius element are in fact pointwise divergence-free (c.f. Remark 2.5.8). This property is highly desirable and its importance has been recognised in recent years; see for instance the discussion in [JLM+17, LR19].

In the finite element formulation of (4.1) we look for $(\mathbf{S},\boldsymbol{u},p) \in \Sigma^n \times V^n \times M^n$ such that

$$\int_\Omega \mathbf{G}(\cdot,\mathbf{S},\mathbf{D}(\boldsymbol{u})) : \boldsymbol{\tau} = 0 \qquad \forall\,\boldsymbol{\tau} \in \Sigma^n, \tag{4.8a}$$

$$\int_\Omega \mathbf{S} : \mathbf{D}(\boldsymbol{v}) - \int_\Omega \boldsymbol{u}\otimes\boldsymbol{u} : \mathbf{D}(\boldsymbol{v}) - \int_\Omega p\,\mathrm{div}\,\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \quad \forall\,\boldsymbol{v} \in V^n, \tag{4.8b}$$

$$-\int_\Omega q\,\mathrm{div}\,\boldsymbol{u} = 0 \qquad \forall\,q \in M^n. \tag{4.8c}$$

The results from Chapter 3 guarantee the existence of solutions of this finite element discretisation and that they converge to a weak solution of (4.1) as the mesh is refined.

The nonlinear system (4.8) is solved using Newton's method. Denoting the current guess for the solution as $(\tilde{\mathbf{S}},\tilde{\boldsymbol{u}},\tilde{p})$, the linearisation procedure is defined by a correction step $(\tilde{\mathbf{S}},\tilde{\boldsymbol{u}},\tilde{p}) \mapsto (\tilde{\mathbf{S}},\tilde{\boldsymbol{u}},\tilde{p}) + (\mathbf{S},\boldsymbol{u},p)$ that is applied iteratively, where $(\mathbf{S},\boldsymbol{u},p)$ is computed by solving a linear system, whose associated matrix presents the following block structure:

$$\begin{bmatrix} Q_1 & Q_2 C^\top & 0 \\ C & E & \tilde{B}^\top \\ 0 & \tilde{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{S} \\ \boldsymbol{u} \\ p \end{bmatrix}. \tag{4.9}$$

The linear operators in the matrix above are defined through the relations

$$\langle Q_1 \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle := \int_\Omega \partial_{\mathbf{S}} \mathbf{G}(\cdot, \tilde{\mathbf{S}}, \mathbf{D}(\tilde{\boldsymbol{u}}))\boldsymbol{\sigma} : \boldsymbol{\tau} \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \Sigma^n, \tag{4.10a}$$

$$\langle Q_2 C^\top \boldsymbol{v}, \boldsymbol{\tau} \rangle := \int_\Omega \partial_{\mathbf{D}} \mathbf{G}(\cdot, \tilde{\mathbf{S}}, \mathbf{D}(\tilde{\boldsymbol{u}}))\mathbf{D}(\boldsymbol{v}) : \boldsymbol{\tau} \qquad \forall\, \boldsymbol{v} \in V^n, \boldsymbol{\tau} \in \Sigma^n, \tag{4.10b}$$

$$\langle \tilde{B} \boldsymbol{v}, q \rangle := -\int_\Omega q \operatorname{div} \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in V^n,\ q \in M^n, \tag{4.10c}$$

$$\langle C \boldsymbol{\sigma}, \boldsymbol{w} \rangle := \int_\Omega \boldsymbol{\sigma} : \mathbf{D}(\boldsymbol{w}) \qquad \forall\, \boldsymbol{\sigma} \in \Sigma^n, \boldsymbol{w} \in V^n, \tag{4.10d}$$

$$\langle E \boldsymbol{v}, \boldsymbol{w} \rangle := -\int_\Omega (\tilde{\boldsymbol{u}} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \tilde{\boldsymbol{u}}) : \mathbf{D}(\boldsymbol{w}) \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in V^n. \tag{4.10e}$$

Note that the differentiability and the strict monotonicity imply, together with the Implicit Function Theorem, that $Q_2^{-1} Q_1$ is either positive or negative definite. If the convective term is neglected (or if Picard linearisation is used instead), with the help of the inf-sup conditions (2.22) and (2.23) we can guarantee that (4.9) is invertible. Although the invertibility of (4.9) is not clear when using Newton's method, in this work we will always employ it, because of its quadratic convergence rate (assuming the current guess is sufficiently close to the solution).

4.1 Augmented Lagrangian preconditioner

4.1.1 Augmented Lagrangian stabilisation

After discretization and Newton linearization, the system has the following block form:

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ p \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ g \end{bmatrix}, \tag{4.11}$$

where $\boldsymbol{z} := (\mathbf{S}, \boldsymbol{u})^\top$, A is the stress-velocity block, and B represents the discrete divergence on the velocity space (c.f. (4.9)). A popular approach to preconditioning systems with this structure is based on the block factorization

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix}^{-1} = \begin{bmatrix} I & -A^{-1}B^\top \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -BA^{-1} & I \end{bmatrix},$$

where $S = -BA^{-1}B^\top$ is the Schur complement. If approximations $\tilde{A}^{-1}$ and $\tilde{S}^{-1}$ of A^{-1} and S^{-1} are available, they can be used in this formula to precondition the coupled system. For a velocity-pressure formulation of the Stokes system, it is known that the Schur complement is spectrally equivalent to the viscosity-weighted pressure mass matrix [SW94, MW11]: $S \sim -\nu^{-1}M_p$. Using (for instance) an algebraic multigrid cycle on A as $\tilde{A}^{-1}$ and the inverse diagonal of the pressure mass matrix as $\tilde{S}^{-1}$ results in a mesh-independent preconditioner for the Stokes system. As mentioned in the Introduction, for the Navier–Stokes system this choice results in a solver whose performance degrades badly as the Reynolds number Re increases, i.e. the number of Krylov iterations per nonlinear iteration grows with Re [ES96]. Other preconditioners such as the pressure convection-diffusion (PCD) [KLW02] and least-squares commutator (LSC) [EHS⁺06] perform well for moderate Reynolds numbers, but their performance still deteriorates as the Reynolds number grows [ESW14].

An alternative approach for dealing with the Schur complement approximation was proposed by Benzi and Olshanskii [BO06] for a 2D Navier–Stokes problem and later extended to the 3D problem by Farrell, Mitchell and Wechsung [FMW19]. The main idea is to modify the system by adding an augmented Lagrangian term:

$$\begin{bmatrix} A + \gamma B^\top M_p^{-1} B & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ p \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} + \gamma B^\top M_p^{-1} g \\ g \end{bmatrix}, \tag{4.12}$$

where $\gamma > 0$ is a parameter. Observe that this modification does not change the solution of the system, since $B\boldsymbol{z} = g$. The addition of the term $\gamma \tilde{B}^\top M_p^{-1} \tilde{B}$ could be interpreted as augmenting the weak formulation with the term

$$\gamma \int_\Omega \Pi_M^n \operatorname{div} \boldsymbol{v} \, \Pi_M^n \operatorname{div} \boldsymbol{w}, \quad \text{for } \boldsymbol{v}, \boldsymbol{w} \in V^n, \tag{4.13}$$

where Π_M^n is the projection to the pressure space M^n. The Scott–Vogelius element satisfies $\operatorname{div}(V^n) \subset M^n$, and so the projection Π_M^n can actually be omitted in this case. The term (4.13) could then be interpreted as a least-squares term that penalizes the L^2-norm of $\operatorname{div} \boldsymbol{u}$, and appears in other contexts, such as the iterated penalty and artificial compressibility methods [Tem68, Cho67]. From the Sherman–Morrison–Woodbury formula (see e.g. [Bac06]) we see that the inverse Schur complement of the augmented matrix can be approximated as

$$\begin{aligned} S^{-1} &= (-B(A + \gamma B^\top M_p^{-1} B)^{-1} B^\top)^{-1} = -(BA^{-1}B^\top)^{-1} - \gamma M_p^{-1} \\ &\approx -(\nu + \gamma) M_p^{-1} \approx -\gamma M_p^{-1}, \end{aligned}$$

with the approximation improving as $\gamma \to \infty$ (cf. [FMW19]).

The challenge is to develop an efficient solver for the augmented top block $A + \gamma B^\top M_p^{-1} B$. This is not trivial as the augmented Lagrangian term has a large kernel (the set of discretely divergence-free velocity fields) and so the matrix degenerates as $\gamma \to \infty$. The essential breakthrough for the Navier–Stokes system came with the work [BO06], where a specialized multigrid operator was developed for the top block, applying ideas developed by Schöberl for nearly incompressible elasticity [Sch99a, Sch99b]. In this work we will apply these ideas to develop a robust multigrid operator for the coupled stress-velocity block. The two main components needed to obtain a robust multigrid solver are a robust relaxation and a robust prolongation operator, which we will develop in the following sections. In the same spirit we mention the work [FMSW20a], where a preconditioner for a Scott–Vogelius discretization of a velocity-pressure formulation of the Newtonian system was developed.

It is important to note that the available theory for the development of robust multigrid solvers assumes that the matrix A is symmetric and positive definite (SPD). This assumption does not hold for the problem under consideration; the stress-velocity block in (4.11) itself has a saddle point structure and is not symmetric. Nevertheless, satisfying the requirements of the SPD case appears to give good performance in the general case also, as observed in the computational experiments of previous works [BO06, FMW19, FMSW20a]. The computational experiments of Section 4.2 demonstrate that the preconditioner we propose possesses similarly excellent robustness with respect to parameters arising in the implicit constitutive relation (4.1b).

As mentioned in Example 2.5.7, barycentric refinement guarantees the inf-sup stability of the Scott–Vogelius element pair for $k \geq d$. However, constructing a multigrid hierarchy by successive barycentric refinement creates degenerate elements. We therefore employ the alternative construction used in [FMSW20a]. The multigrid hierarchy is obtained by taking a standard uniformly-refined hierarchy and barycentrically refining on each level once; see Figure 4.2. The cells before barycentric refinement are referred to as *macro cells*. An important consequence of this is the existence of local Fortin operators on each macro cell, which are useful when trying to characterize locally the space of divergence-free velocities [FMSW20b]. A disadvantage is that the resulting mesh hierarchy is non-nested, which leads to some complications with the prolongation operator in the multigrid algorithm.

Remark 4.1.1. *Augmented Lagrangian preconditioners have been applied to flow problems with variable viscosity and Bingham rheology before; see e.g. [HNV15, HN12].*

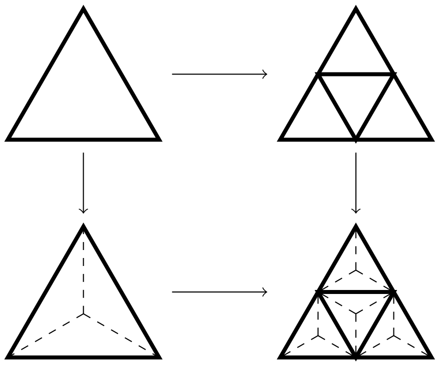


Figure 4.2: Non-nested two-level barycentrically refined mesh hierarchy in two dimensions.

In those works it is advocated that for the Schur complement approximation a viscosity-weighted mass matrix should be used instead:

$$(M_\mu)_{ij} := \int_\Omega \frac{1}{\mu}\phi_i\, \phi_j, \tag{4.14}$$

where μ denotes the variable (effective) viscosity and ϕ_i, ϕ_j are pressure basis functions. A similar argument was presented in [GO09], where only the Schur complement approximation without the augmented Lagrangian term was studied. However, in those works a robust scalable solver for the augmented momentum block was not available and the authors were limited to low values of γ ($\gamma = 1$ was used in their numerical experiments), and so a better approximation for the Schur complement with (4.14) was necessary. In contrast, the multigrid solver presented in this work for the stress-velocity block will be γ-robust, which therefore allows for very large values of γ, and thus excellent control of the Schur complement. It is consequently not necessary to use (4.14), which requires reassembly at every Newton step.

4.1.2 Solving the top block: robust relaxation

From (4.12) and (4.9), we see that the augmented stress-velocity block can be written as

$$A_{n,\gamma} := A + \gamma B^\top M_p^{-1} B = \begin{bmatrix} Q_1 & Q_2 C^\top \\ C & E \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ \tilde{B}^\top \end{bmatrix} M_p^{-1} \begin{bmatrix} 0 & \tilde{B} \end{bmatrix}, \tag{4.15}$$

where $\gamma B^\top M_p^{-1} B$ is symmetric and semi-definite and A is invertible. Relaxation methods used in multigrid algorithms can often be framed in terms of subspace correction methods [Xu92, Xu01]. Let us define $Z^n := \Sigma^n \times V^n$ and consider the subspace decomposition

$$Z^n = \sum_i Z_i, \tag{4.16}$$

where the sum is not necessarily direct. Denoting the natural inclusion by $I_i \colon Z_i \to Z^n$, we can define the restriction of $A_{n,\gamma}$ to Z_i as

$$\langle A_i \boldsymbol{z}_i, \boldsymbol{w}_i \rangle := \langle A_{n,\gamma} I_i \boldsymbol{z}_i, I_i \boldsymbol{w}_i \rangle, \quad \forall\, \boldsymbol{z}_i, \boldsymbol{w}_i \in Z_i. \tag{4.17}$$

The parallel subspace correction method, or additive Schwarz preconditioner, associated with (4.16) is then defined by the action

$$D_{n,\gamma}^{-1} = \sum_i I_i A_i^{-1} I_i^*. \tag{4.18}$$

The multiplicative version, or sequential correction method, is similar, but applies the correction step on each space sequentially. For instance, the Jacobi and Gauss-Seidel smoothers are obtained by setting $Z_i = \operatorname{span}\{\boldsymbol{\varphi}_i\}$, where $\{\boldsymbol{\varphi}_i\}_i$ denotes a basis of Z^n, and using the parallel and sequential correction methods, respectively.

When $A_{n,\gamma}$ is symmetric and positive definite (e.g. as with a velocity-pressure formulation of the Stokes system), one can investigate whether $D_{n,\gamma}$ and $A_{n,\gamma}$ are spectrally equivalent, in order to study the effectiveness of the preconditioner defined by (4.18). A useful fact in that case is that the square of the norm induced by $D_{n,\gamma}$ can be written as [Xu01]:

$$\|\boldsymbol{v}\|_{D_{n,\gamma}}^2 = \inf_{\substack{\boldsymbol{v}_i \in Z_i \\ \sum_i \boldsymbol{v}_i = \boldsymbol{v}}} \|\boldsymbol{v}_i\|_{A_i}^2. \tag{4.19}$$

For example, for the Jacobi smoother we can obtain the bound (see [FMSW20b])

$$\|\boldsymbol{v}\|_{D_{n,\gamma}}^2 = \sum_i \|\boldsymbol{v}\|_{A_{n,\gamma}}^2 \le c\frac{1+\gamma}{h_n^2} \sum_i \|\boldsymbol{v}\|_{L^2(\Omega)}^2 \le c(1+\gamma) h_n^{-2} \|\boldsymbol{v}\|_{A_{n,\gamma}}^2, \tag{4.20}$$

where c is independent of n and γ. The estimate (4.20) degenerates as the mesh is refined and this explains why it is usually not effective as a standalone solver, and so it must be used in a multigrid hierarchy. In addition, the estimate also degenerates as γ increases, so the preconditioner will not be robust with respect to γ. The following result states the conditions needed to guarantee that the subspace correction method is parameter-robust (a similar result for the multiplicative method can be found in [LWXZ07]).

Theorem 4.1.2 ([Sch99b, FMSW20b])**.** *Let $\{Z_i\}_i$ be a subspace decomposition of Z^n and denote the maximal number of overlapping subspaces of any one subspace by N_O. Assume that A is symmetric and positive definite and denote the kernel by*

$$\mathcal{N}^n = \{\boldsymbol{z} \in Z^n \colon \tilde{B}\boldsymbol{z} = 0\}. \tag{4.21}$$

Assume that the pair $Z^n \times M^n$ is inf-sup stable for the problem associated to the bilinear form

$$\langle \bar{B}(\boldsymbol{z}, p), (\boldsymbol{w}, q)\rangle := \langle A\boldsymbol{z}, \boldsymbol{w}\rangle - \langle \tilde{B}\boldsymbol{w}, p\rangle - \langle \tilde{B}\boldsymbol{z}, q\rangle, \tag{4.22}$$

and that any $\boldsymbol{v} \in Z^n$ and $\boldsymbol{v}_0 \in \mathcal{N}^n$ satisfy

$$\begin{aligned} \inf_{\substack{\boldsymbol{z}=\sum_i \boldsymbol{z}_i \\ \boldsymbol{z}_i \in Z_i}} \sum_i \|\boldsymbol{z}_i\|^2_{H^1(\Omega)} &\leq c_1(n)\|\boldsymbol{z}\|^2_{L^2(\Omega)}, \\ \inf_{\substack{\boldsymbol{z}_0=\sum_i \boldsymbol{z}_{0,i} \\ \boldsymbol{z}_{0,i} \in \mathcal{N}^n \cap Z_i}} \sum_i \|\boldsymbol{z}_{0,i}\|^2_{H^1(\Omega)} &\leq c_2(n)\|\boldsymbol{z}_0\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.23}$$

Then the following holds for $\hat{c}, \tilde{c} > 0$ independent of n and γ:

$$\hat{c}(c_1(n) + c_2(n))^{-1} D_{n,\gamma} \leq A_{n,\gamma} \leq \tilde{c} N_O D_{n,\gamma}. \tag{4.24}$$

For the system (4.15), the kernel $\mathcal{N}^n$ consists of all the vectors of the form $(\boldsymbol{\sigma}, \boldsymbol{u})^\top$, where $\boldsymbol{u} \in V^n_{\text{div}}$ and $\boldsymbol{\sigma} \in \Sigma^n$ is arbitrary. An implicit assumption in Theorem 4.1.2 is that the spaces in the decomposition (4.16) are sufficiently rich to capture the kernel:

$$\mathcal{N}^n = \sum_i Z_i \cap \mathcal{N}^n. \tag{4.25}$$

Thankfully, a local characterisation of the kernel of the divergence operator for the Scott–Vogelius discretisation on meshes with the macro element structure considered here was recently obtained in [Wec19] (see also [FMSW20b]). In that work it was proved that a kernel capturing space decomposition is obtained by setting

$$Z_i := \{\boldsymbol{v} \in Z^n \ : \ \operatorname{supp}(\boldsymbol{v}) \subset \operatorname{macrostar}(q_i)\}, \tag{4.26}$$

where for each vertex q_i, the macrostar patch macrostar(q_i) is defined as the union of all macro elements touching the vertex (Figure 4.3 shows a two-dimensional example).

Proposition 4.1.3 ([Wec19, FMSW20b])**.** *Assume that Ω is simply connected and consider the subspace decomposition defined by (4.26). Then, for any $\boldsymbol{z} \in Z^n$ and $\boldsymbol{z}_0 \in \mathcal{N}^n$ we have that*

$$\begin{aligned} \inf_{\substack{\boldsymbol{z}=\sum_i \boldsymbol{z}_i \\ \boldsymbol{z}_i \in Z_i}} \sum_i \|\boldsymbol{z}_i\|^2_{H^1(\Omega)} &\leq c_1(n)\|\boldsymbol{z}\|^2_{L^2(\Omega)}, \\ \inf_{\substack{\boldsymbol{z}_0=\sum_i \boldsymbol{z}_{0,i} \\ \boldsymbol{z}_{0,i} \in \mathcal{N}^n \cap Z_i}} \sum_i \|\boldsymbol{z}_{0,i}\|^2_{H^1(\Omega)} &\leq c_2(n)\|\boldsymbol{z}_0\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.27}$$

Consequently, the following holds for $\hat{c}, \tilde{c} > 0$ independent of n and γ:

$$\hat{c}(c_1(n) + c_2(n))^{-1} D_{n,\gamma} \leq A_{n,\gamma} \leq \tilde{c} N_O D_{n,\gamma}. \tag{4.28}$$

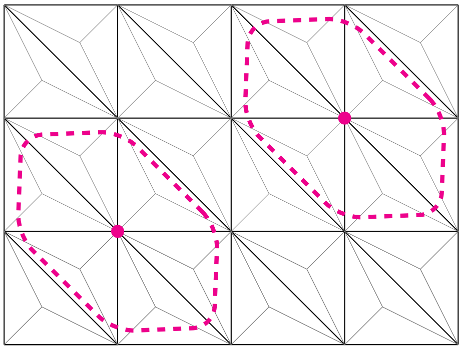


Figure 4.3: Macrostar patches on a barycentrically refined mesh in two dimensions.

Remark 4.1.4. *In some cases the analysis can be carried out in a slightly different manner. For example, if we take a Bercovier–Engelman-like regularization of the constitutive relation for an activated Euler fluid (this is the counterpart of the Bingham constitutive relation where the roles of* S *and* D *are interchanged, see e.g. [BMR20])*

$$\mathsf{G}(\mathsf{S}, \mathsf{D}) = \mathsf{D} - \left(\frac{1}{2\nu} + \frac{\tau_y}{\sqrt{\varepsilon^2 + |\mathsf{S}|^2}}\right)\mathsf{S},$$

with $\nu, \varepsilon > 0$ *and* $\tau_y \geq 0$*, then the stress-velocity block in the linearised problem can be split as follows:*

$$\hat{A}_\nu + \hat{A}_\varepsilon + \gamma B^\top M_p^{-1} B, \tag{4.29}$$

where $\hat{A}_\nu$ *corresponds to the operator arising from the Newtonian problem and* $\hat{A}_\varepsilon$ *is defined via*

$$\langle \hat{A}_\varepsilon(\boldsymbol{\sigma}, \boldsymbol{v}), (\boldsymbol{\tau}, \boldsymbol{w})\rangle := \tau_y \int_\Omega \frac{1}{\sqrt{\varepsilon^2 + |\tilde{\mathsf{S}}|^2}} \left[\mathsf{I} - \frac{\tilde{\mathsf{S}} \otimes \tilde{\mathsf{S}}}{\varepsilon^2 + |\tilde{\mathsf{S}}|^2}\right] \boldsymbol{\sigma} : \boldsymbol{\tau}, \quad \forall (\boldsymbol{\sigma}, \boldsymbol{v}), (\boldsymbol{\tau}, \boldsymbol{w}) \in Z^h.$$

The splitting (4.29) could then be interpreted as a perturbation of the Newtonian problem, which results in an operator that degenerates as $\varepsilon \to 0$*,* $\gamma \to \infty$*, with a kernel given by elements of the form* $(\tilde{\mathsf{S}}, \boldsymbol{w}) \in Z^h$*, with* $\operatorname{div} \boldsymbol{w} = 0$*. Note that while the kernel possesses a one-dimensional stress component, in practice this does not appear to cause any difficulties for the preconditioner. An illustrative example for a slightly more complicated problem will be shown in the final section of this work.*

In the algorithm presented here, the relaxation solves will be performed additively. For the patches depicted in Figure 4.3, each coupled stress-velocity solve for $k = 2$ (resp. $k = 3$) involves 31 (resp. 73) degrees of freedom for each component of the velocity and 60 (resp. 156) degrees of freedom for the stress. This is much more expensive than, say, a Jacobi smoother, but the resulting robustness in the algorithm makes it worth the cost, and small local patchwise solves are quite well suited to modern computing architectures.

Remark 4.1.5. *When working with the full nonlinear problem including advection, the macrostar iteration* (4.16) *&* (4.26) *is not effective as a standalone relaxation method. However, as observed in [FMW19, FMSW20a], this difficulty can be overcome by applying a small number of GMRES iterations preconditioned by the macrostar iteration as relaxation.*

4.1.3 Solving the top block: robust prolongation

As illustrated in [FMW19, FMSW20b], a robust multigrid algorithm also requires a stable prolongation operator $P_N : V^N \to V^n$, mapping the space of coarse grid functions V^N into the space of fine grid functions V^n, with a continuity constant independent of γ. For the Stokes problem, the matrix A acts only on the velocity space V^n and is actually SPD and thus the whole matrix (4.15) defines a norm. We could therefore write:

$$\|\boldsymbol{v}_N\|_{N,\gamma}^2 = \|\boldsymbol{v}_N\|_{A_N}^2 + \gamma\|\operatorname{div}\boldsymbol{v}_N\|_{L^2(\Omega)}^2,$$
$$\|P_N\boldsymbol{v}_N\|_{n,\gamma}^2 = \|P_N\boldsymbol{v}_N\|_{A_n}^2 + \gamma\|\operatorname{div}(P_N\boldsymbol{v}_N)\|_{L^2(\Omega)}^2,$$

where A_N and A_n correspond to discretisations on the coarse and fine mesh, respectively. The central difficulty is that the condition $\operatorname{div}\boldsymbol{v}_N = 0$ does not necessarily imply that $\operatorname{div}(P_N\boldsymbol{v}_N) = 0$, when P_N is a standard prolongation operator based on finite element interpolation, due to the non-nestedness of the mesh hierarchy. If not addressed, this causes a lack of robustness in the multigrid solver for large γ. The insight of Schöberl [Sch99a, Sch99b], later applied by Benzi and Olshanskii in [BO06], and Farrell, Mitchell and Wechsung [FMW19, FMSW20b, FMSW20a], is that by performing local Stokes solves it is possible to compute a correction to the prolongation operator and ensure that divergence-free fields get mapped to (nearly) divergence-free fields.

Proposition 4.1.6 ([Wec19, FMSW20b])**.** *Assume we can split $M^n = \tilde{M}^N \oplus \hat{M}^n$, with $\tilde{M}^N \subset M^N$. Let $P_N \colon V^N \to V^n$ be a prolongation operator that is continuous in the $\|\cdot\|_{W^{1,2}(\Omega)}$ norm and that preserves the divergence with respect to $\tilde{M}^N$, i.e.*

$$\int_\Omega \operatorname{div}(P_N\boldsymbol{v}_N)\tilde{q}_N = \int_\Omega \operatorname{div}\boldsymbol{v}_N\tilde{q}_N \quad \forall\, \tilde{q}_N \in \tilde{M}^N, \boldsymbol{v}_N \in V^N. \tag{4.30}$$

Suppose further that there exists $\hat{V}^n \subset V^n$ such that the pair $\hat{V}^n \times \hat{M}^n$ is inf-sup stable and that

$$\int_\Omega \tilde{q}_N \operatorname{div} \hat{\boldsymbol{v}}_n = 0 \quad \forall\, \tilde{q}_N \in \tilde{M}^N, \hat{\boldsymbol{v}}_n \in \hat{V}^n, \tag{4.31}$$

Given $\boldsymbol{v}_N \in V_N$, define $\tilde{\boldsymbol{v}}_n$ as the solution of the problem

$$\langle A_{n,\gamma} \tilde{\boldsymbol{v}}_n, \hat{\boldsymbol{w}}_n \rangle = \gamma \int_\Omega \operatorname{div}(P_N \boldsymbol{v}_N) \operatorname{div}(\hat{\boldsymbol{w}}_n) \quad \forall\, \hat{\boldsymbol{w}}_n \in \hat{V}^n. \tag{4.32}$$

Then the prolongation operator $\tilde{P}_N \colon V^N \to V^n$ defined by

$$\tilde{P}_N \boldsymbol{v}_N := P_N \boldsymbol{v}_N - \tilde{\boldsymbol{v}}_n, \tag{4.33}$$

is continuous in the energy norm.

For the Scott–Vogelius discretisation on meshes with the macro element structure shown in Figure 4.2, the interpolation is actually exact on the boundaries of the macro cells, and therefore, as shown in [FMSW20b, FMSW20a], the divergence is preserved with respect to the space

$$\tilde{M}^N := \{q \in L^2(\Omega) \colon q \text{ is constant on } K \in \mathcal{M}_N\}, \tag{4.34}$$

where $\mathcal{M}_N$ is the triangulation of coarse macro elements. Consequently, the following choice of spaces satisfies the requirements of Proposition 4.1.6 (see [FMSW20b]):

$$\begin{aligned} \hat{M}^n &:= \{q_n \in M^n \colon \Pi_{\tilde{M}^N} q_n = 0\}, \\ \hat{V}^n &:= \{\boldsymbol{v}_n \in V_n \colon \operatorname{supp}(\boldsymbol{v}_n) \subset K \text{ for some } K \in \mathcal{M}_N\}, \end{aligned} \tag{4.35}$$

where $\Pi_{\tilde{M}^N}$ denotes the orthogonal projection onto $\tilde{M}^N$, and hence the prolongation defined by (4.33) will be robust in γ. Observe that, by definition of the space $\tilde{V}^h$, the problem (4.32) decouples on the patches defined by the macro elements and can therefore be computed independently on each macro cell (see Figure 4.4); this is extremely important for the efficiency of the solver.

In the non-Newtonian setting, in the computation of the modified prolongation operator it may seem more appropriate to alternatively employ on the left-hand side of (4.32) the operator defined by the Schur complement $-CQ_1^{-1}Q_2C^\top$ (which reduces to (4.32) in the Newtonian case). However, since the end-goal is to correct for the error in the divergence introduced by the interpolation operator, we prefer to retain (4.32) for the sake of simplicity and avoiding reassembly and refactorisation.

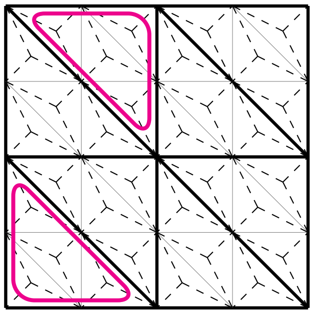


Figure 4.4: The correction to the prolongation operator is computed on the coarse macro cells.

The prolongation operator for the stress variable $\boldsymbol{\sigma}_N \in \Sigma^N \mapsto \boldsymbol{\sigma}_n \in \Sigma^n$, between spaces Σ^N and Σ^n defined on the coarse and fine meshes respectively, is defined via the Galerkin projection

$$\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_N\|_{L^2(\Omega)} = \min_{\boldsymbol{\sigma}\in\Sigma^n} \|\boldsymbol{\sigma}_N - \boldsymbol{\sigma}\|_{L^2(\Omega)}. \tag{4.36}$$

If we denote the basis of Σ^n by $\{\boldsymbol{\varphi}_n^i\}_{i=1}^{N_n}$, then the optimality condition for (4.36) takes the form

$$\int_\Omega \boldsymbol{\sigma}_n : \boldsymbol{\varphi}_n^i = \int_\Omega \boldsymbol{\sigma}_H : \boldsymbol{\varphi}_n^i \quad \forall\, i \in \{1, \ldots, N_n\}, \tag{4.37}$$

or written in matrix form:

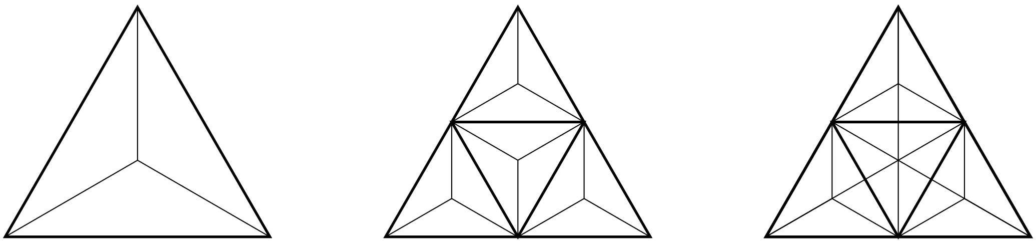
$$M_n \boldsymbol{\sigma}_n = M_{n,N} \boldsymbol{\sigma}_N, \tag{4.38}$$

where the mass matrices are defined as

$$\begin{aligned} (M_n)_{ij} &= \int_\Omega \boldsymbol{\varphi}_n^i : \boldsymbol{\varphi}_n^j \quad i, j \in \{1, \ldots, N_n\}, \\ (M_{n,N})_{ij} &= \int_\Omega \boldsymbol{\varphi}_n^i : \boldsymbol{\varphi}_N^j \quad i \in \{1, \ldots, N_n\},\, j \in \{1, \ldots, N_N\}, \end{aligned} \tag{4.39}$$

where the basis of Σ^N is denoted by $\{\boldsymbol{\varphi}_N^i\}_{i=1}^{N_N}$.

Since the meshes are non-nested, the assembly of $M_{n,N}$ requires the integration of *piecewise* polynomial functions over the cells of either mesh. To integrate these accurately we construct a *supermesh* of both input meshes [FM11], a common refinement of both (see Figure 4.5). Over each supermesh cell the integrand of the right-hand side of (4.37) is polynomial, and hence can be calculated accurately with standard quadrature rules. Since the stress is approximated using discontinuous piecewise polynomials, the mass matrix M_n is block diagonal, and is simple to invert exactly.



(a) Coarse mesh $\mathcal{T}_N$. (b) Fine mesh $\mathcal{T}_n$. (c) A supermesh of $\mathcal{T}_H$ and $\mathcal{T}_h$.

Figure 4.5: Example of a coarse mesh $\mathcal{T}_N$, a fine mesh $\mathcal{T}_n$, and an associated supermesh.

As rediscretization is employed to assemble coarse grid problems, the current guess for the stress must be injected onto coarse grids. Injection is defined via a Galerkin projection analogous to (4.36), and employs the same supermesh.

The theory developed in [LWXZ07] and [Sch99a] assumes that the operator A is symmetric and positive definite, and therefore it does not cover the 3-field formulation (4.9) since the top block has a saddle point structure and is non-symmetric due to the convective term. However, we observe that the same strategy of using kernel capturing space decompositions and a corrected prolongation operator yields preconditioners with the same good qualities as in the symmetric and definite case. An overview of the full algorithm can be found in Figure 4.6.

Remark 4.1.7. *In the Stokes problem the operator appearing in the top block in* (4.9) *reduces after augmentation to*

$$\begin{bmatrix} -\frac{1}{2\nu}\mathsf{I} & C^\top \\ C & 0 \end{bmatrix} + \gamma B^\top M_p^{-1} B. \tag{4.40}$$

Note that if written in terms of matrices, the identity operator I *becomes a mass matrix for the space* Σ^n*. In this case the operator* (4.40) *also degenerates as* $\nu \to 0$ *or* $\gamma \to \infty$*, but now it is not possible to perform the analysis in terms of a single parameter* $\hat{\gamma} = \gamma/\nu$ *(as was done in [Wec19]), since the limits now have a different character. As* $\gamma \to \infty$*, the augmented Lagrangian term dominates and therefore its kernel will play a role in the analysis; on the other hand, as* $\nu \to 0$ *the stress block becomes instead the dominant term, and so it is the invertibility of the whole saddle point operator* A *that suffers. This suggests that in order to extend the results of [Sch99b, LWXZ07] to the case of saddle point matrices, a new approach has to be found.*

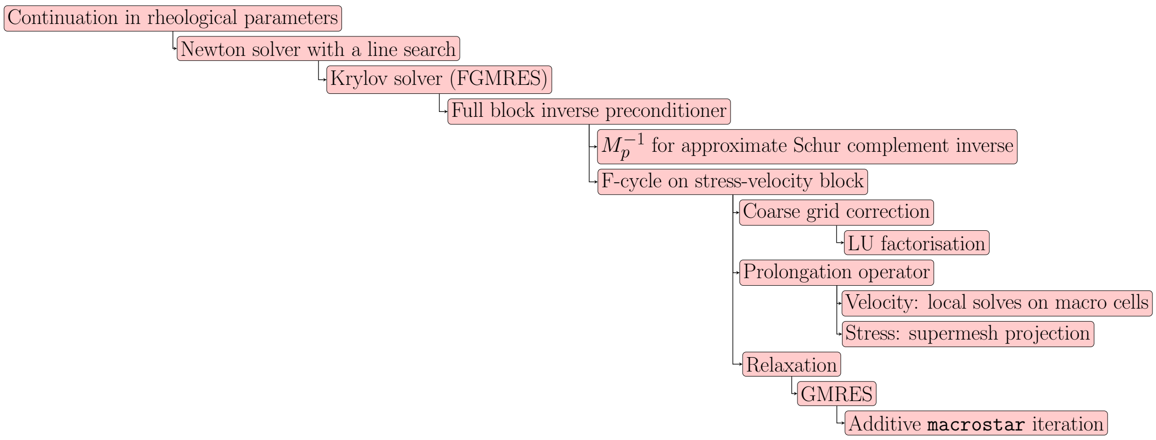


Figure 4.6: Overview of the algorithm.

Remark 4.1.8. *If the constitutive relation can be written in the form* $\mathsf{G}(\cdot, \mathsf{S}, \mathsf{D}) = \tilde{\alpha}(\cdot, |\mathsf{D}|^2)\mathsf{D} - \mathsf{S}$*, then the shear stress* S *can be eliminated from the system and a velocity-pressure formulation can be obtained. In this case the theory of [LWXZ07, Sch99a, FMSW20b] does apply, assuming the convective term is neglected. One potential difficulty, if several terms appear because of the non-Newtonian constitutive relation, is that the different kernels could interact in such a way that this affects the properties of the algorithm, but in practice this was not observed to be the case.*

4.2 Numerical examples

All the numerical examples presented in this work were implemented using Firedrake [RHM+16]. The macrostar patch solves for the relaxation and the local solves for the prolongation operator in the multigrid algorithm were carried out with PCPATCH [FKMW19], a recently developed preconditioner in PETSc [BAA+17] for matrix-free multigrid relaxation via space decompositions. The L^2 line search algorithm [BKST15] was employed to improve the convergence of the Newton solver; the Newton solver was deemed to have converged when the Euclidean norm of the residual fell below 1×10^{-8} and the corresponding tolerance for the linear solver was set to 1×10^{-10}, unless specified otherwise. These tight tolerances are taken to challenge the solver; in practical computations the tolerance on the linear solver should be dynamically adjusted to minimise the computational work, using e.g. the Eisenstat–Walker algorithm [EW96]. The augmented Lagrangian parameter was taken as $\gamma = 10^4$, to obtain excellent control of the Schur complement. In the implementation, the uniqueness of the pressure was recovered not by enforcing a zero mean condition in the variational formulation but rather by orthogonalizing against the nullspace of constants in the Krylov solver.

4.2.1 Bingham flow between two plates

We first test our solver on a problem where the exact solution is known. Let $\Omega = (0, L) \times (-1, 1)$ with $L > 0$ and consider problem (4.1) with $\boldsymbol{f} = \boldsymbol{0}$ and the Bingham constitutive relation (4.5). A function that solves this problem exactly is given by [AHOV11, GO09, HMST17]:

$$\boldsymbol{u}_e(\boldsymbol{x}) := \begin{cases} (\frac{C}{2}(1-x_2^2) - \tau_y(1-x_2), 0)^\top, & \text{if } \frac{\tau_y}{C} \leq x_2 \leq 1, \\ (\frac{C}{2}(1-(\frac{\tau_y}{C})^2) - \tau_y(1-\frac{\tau_y}{C}), 0)^\top, & \text{if } -\frac{\tau_y}{C} \leq x_2 \leq \frac{\tau_y}{C}, \\ (\frac{C}{2}(1-x_2^2) - \tau_y(1+x_2), 0)^\top, & \text{if } -1 \leq x_2 \leq -\frac{\tau_y}{C}, \end{cases} \tag{4.41a}$$

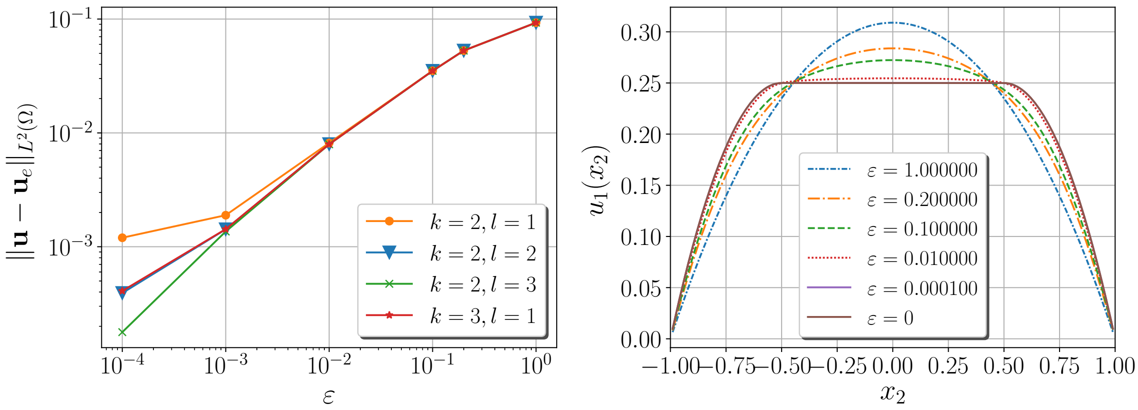
$$p_e(\boldsymbol{x}) := -C(x_1 - \frac{L}{2}), \tag{4.41b}$$

where C is the (negative) pressure gradient. The boundary datum $\boldsymbol{u}_b$ is chosen so as to match the values in the expression above. The problem was solved with $L = 4$, $C = 2$ and $\tau_y = 1$ using the regularisation (4.7a). Secant continuation starting from $\varepsilon = 1$ was employed to obtain better initial guesses for Newton's method; more precisely, this means that given two previously computed solutions $\boldsymbol{w}_1, \boldsymbol{w}_2$ corresponding to the parameters $\varepsilon_1, \varepsilon_2$, respectively, the initial guess for Newton's method at ε is chosen as

$$\frac{\varepsilon - \varepsilon_2}{\varepsilon_2 - \varepsilon_1}(\boldsymbol{w}_2 - \boldsymbol{w}_1) + \boldsymbol{w}_2. \tag{4.42}$$

In this case the tolerances were chosen to be 1×10^{-10} and 1×10^{-12} for the nonlinear and linear solvers, respectively. Tighter tolerances are used for this problem to ensure convergence of the continuation scheme.

Figure 4.7 (a) shows the L^2-distance between the numerical solution and the exact solution (4.41), as ε decreases, for different values of the polynomial degree k and the number of refinements in the mesh hierarchy l; it can be observed that at some point the discretization error starts to dominate. Figure 4.7 (b) shows the velocity profiles for different values of ε, including the exact solution. Table 4.1 shows the average number of Krylov iterations per Newton step using two multigrid cycles with 5 relaxation sweeps per level as $\tilde{A}^{-1}$. It can be seen in Table 4.1 that the number of iterations remains under control, with only a slight increase for very small ε and one level of refinement; the number of Newton iterations also appears to exhibit mesh-independence. In the practical computation of viscoplastic flow the approach



(a) Errors for different values of polynomial degree k and number of levels l.

(b) Velocity profiles at $x_1 = 2$ for different values of ε, including the exact solution $\varepsilon = 0$.

Figure 4.7: Numerical solution of the Bingham flow between two plates.

described here should be combined with an adaptive refinement of the mesh in order to resolve the yield surface more accurately.

k	# refs	# dofs	ε			
			0.1	0.01	0.001	0.0001
	1	2.8×10^4	5	5	5.33	14
2	2	1.1×10^5	4	3.57	3.83	2.66
	3	4.5×10^5	4	4	3.85	3.5
3	1	5.9×10^4	2.4	2.6	2.44	3.5

Table 4.1: Average number of Krylov iterations per Newton step as ε decreases for the Bingham flow between two plates.

4.2.2 Generalised Carreau–Yasuda fluid

In this experiment we employ the constitutive relation (4.4) and test the solver with different values of the rheological parameters on the lid driven cavity problem. The problem is solved on the square/cube $(0,2)^d$ with $\boldsymbol{f} = \mathbf{0}$, and boundary data

$$\boldsymbol{u}_b(\boldsymbol{x}) := \begin{cases} (x^2(2-x)^2, 0)^\top, & \text{if } y = 2, \\ (0,0)^\top, & \text{otherwise,} \end{cases}$$

if $d = 2$, and

$$\boldsymbol{u}_b(\boldsymbol{x}) := \begin{cases} (x^2(2-x)^2 z^2(2-z)^2, 0, 0)^\top, & \text{if } y = 2, \\ (0,0,0)^\top, & \text{otherwise,} \end{cases}$$

k	# refs	# dofs	ν			
			0.2	0.001	0.0005	0.0002
	1	3.1×10^4	4.25	3.5	4	5
2	2	1.2×10^5	4.25	3.5	3.5	4
	3	4.9×10^5	4.25	3	2.5	3
	1	6.5×10^4	2.75	2.	2.5	2.5
3	2	2.5×10^5	2.75	1.66	2	2.5
	3	1.0×10^6	2.5	2	1.5	1.5

Table 4.2: Average number of Krylov iterations per Newton step as ν decreases for the 2D generalised Carreau–Yasuda relation with $r_1 = 1.8$, $r_2 = 2.5$, $\Gamma_1 = \Gamma_2 = 200$, $\beta_1 = 0.9$, $\beta_2 = 0.5$.

if $d = 3$. For the 3D problem the tolerance for the linear solver was set to 1×10^{-8}. In this example a simple continuation algorithm was employed to reach the different values of the parameters, e.g. the solution corresponding to ν is used as an initial guess in Newton's method for the problem with $\nu + \delta\nu$, iterating the procedure until the desired value is reached. For parameters for which the effective viscosity is small (e.g. small ν), the problem will be convection dominated and hence some advective stabilisation is required in (4.8). We choose to add a stabilising term based on jump penalisation described in [BL08, DD76]:

$$S_h(\boldsymbol{v}, \boldsymbol{w}) := \sum_{K \in \mathcal{M}_h} \frac{1}{2} \int_{\partial K} \delta\, h_{\partial K}^2 \, [\![\nabla \boldsymbol{v}]\!] : [\![\nabla \boldsymbol{w}]\!], \tag{4.43}$$

where $[\![\boldsymbol{z}]\!]$ denotes the jump of $\boldsymbol{z}$ across ∂K, $h_{\partial K}$ is a function giving the size of each face in ∂K, and δ is an arbitrary stabilisation parameter. In the numerical experiments the stabilisation parameter was chosen to be cell-dependent and set to $5 \times 10^{-3} \|\tilde{u}\|_{L^\infty(K)}$. In the experiments described in this section, 2 full multigrid cycles with 4 relaxation sweeps per level were applied as $\tilde{A}^{-1}$ when $d = 2$, and 1 cycle with 6 relaxation sweeps when $d = 3$. These values were chosen so as to balance the amount of inner and outer work (e.g. fewer relaxation sweeps result in less expensive linear solves, but more iterations are needed); convergence is also achieved with fewer relaxation sweeps, but the values chosen here resulted in a shorter time to solution. Tables 4.2 and 4.3 show the average number of Krylov iterations per Newton step for a problem with decreasing ν; it can be observed that the number of iterations remains well controlled even for the lowest values of ν (in the Newtonian problem, $\nu = 0.0002$ would correspond to a Reynolds number of 10000).

k	# refs	# dofs	ν 0.2	0.002	0.0005	0.00028
3	1	9.2×10^5	7.25	5	5.5	5.5

Table 4.3: Average number of Krylov iterations per Newton step as ν decreases for the 3D generalised Carreau–Yasuda relation with $r_1 = 1.8$, $r_2 = 2.5$, $\Gamma_1 = \Gamma_2 = 200$, $\beta_1 = 0.9$, $\beta_2 = 0.5$.

A comparison with the preconditioner using a Jacobi smoother instead of the macrostar iteration can be found in Table 4.4, for a given set of rheological parameters. The experiments were performed on 12 Intel Xeon Silver 4116 CPUs. Very mild parameters are considered for this comparison, since the Krylov solver with Jacobi smoothing fails to converge otherwise (the solver using the AMG libraries Hypre [FY02], ML [GSH+06], and GAMG [ABKP04] on the stress-velocity block failed to converge altogether). We note that for our academic test problems, the preconditioner employing a direct sparse solver for the stress-velocity block is still faster on the workstation resources we had available, but we expect that the implementation could be optimised and the algorithm employing the macrostar iteration will scale better on high performance computers. Other ways of lowering the cost of the algorithm, such as employing $H(\mathrm{div})$–L^2-type elements for the velocity and pressure, for which a smaller star iteration would suffice to capture the kernel, will be the subject of future research.

d	$(k, \#\text{refs})$	# dofs	macrostar # iters	macrostar time (min.)	Jacobi # iters	Jacobi time (min.)
2	$(2, 3)$	4.9×10^5	15	1.67	3040	107.81
3	$(3, 1)$	9.2×10^5	10	37.33	753	70.35

Table 4.4: Runtime comparison and total number of Krylov iterations (# iters) for a 2D problem with $\nu = 0.02$, $r_1 = 1.8$, $r_2 = 2.5$, $\Gamma_1 = \Gamma_2 = 200$, $\beta_1 = 0.9$, $\beta_2 = 0.5$, and a 3D problem with $r_1 = r_2 = \nu = 2$.

Tables 4.5 and 4.6 show the number of average Krylov iterations for small r_1 and large Γ_2, respectively, for two different values of γ. It can be observed that depending on the parameter of interest, large values of γ improve the robustness of the algorithm. In all the examples the solver appears to be robust with respect to the parameters appearing in the constitutive relation and also exhibits mesh-independence.

γ	k	# refs	# dofs	r_1			
				1.66	1.25	1.11	1.07
10^4	2	1	3.1×10^4	3.5	3.5	3.5	3.5
		2	1.2×10^5	3.5	3.5	3.5	3.5
		3	4.9×10^5	3	3.5	4	4
	3	1	6.5×10^4	2	2	2	2
		2	2.5×10^5	2	2	2	2.5
		3	1.0×10^6	2	2	2.5	2.5
1	2	1	3.1×10^4	5	4	4	4
		2	1.2×10^5	4.5	4	3.5	3.5
		3	4.9×10^5	4	4	4	4
	3	1	6.5×10^4	4	4	3.5	3
		2	2.5×10^5	4	3.5	3	3
		3	1.0×10^6	4	3.5	3	3

Table 4.5: Average number of Krylov iterations per Newton step as r_1 decreases for the 2D generalised Carreau–Yasuda relation with $\nu = 0.01$, $r_2 = 2$, $\Gamma_1 = 125$, $\beta_1 = 0.7$.

Remark 4.2.1. *In general, extreme values of the parameters could result in convergence issues for the nonlinear iterations. In practice, the preconditioner presented here should then be coupled e.g. with a more sophisticated continuation strategy for the nonlinear iterations, or with nested iteration.*

4.2.3 Activated Euler Flow Past an Obstacle

Consider the constitutive relation of an Euler/power-law fluid (c.f. (3.41)):

$$\begin{cases} \mathbf{D} = \tau_y \frac{\mathbf{S}}{|\mathbf{S}|} + \mathbf{D}_2, & \text{if } |\mathbf{D}| \geq \tau_y, \\ \mathbf{S} = 0, & \text{if } |\mathbf{D}| < \tau_y, \end{cases} \tag{4.44}$$

where $\mathbf{D}_2$ satisfies $\mathbf{S} = 2\nu|\mathbf{D}_2|^{r-2}\mathbf{D}_2$, for some $\nu > 0$ and $r > 1$. Observe that the power-law nonlinearity can be inverted and we have that, for any $\mathbf{D}, \mathbf{S} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$,

$$\mathbf{S} = 2\nu|\mathbf{D}|^{r-2}\mathbf{D} \Longleftrightarrow \mathbf{D} = \frac{1}{2\nu}\left(\frac{|\mathbf{S}|}{2\nu}\right)^{r'-2}\mathbf{S}. \tag{4.45}$$

Using this fact we can write a regularized constitutive relation similar to the one described in Remark 4.1.4:

$$\mathbf{G}(\mathbf{S}, \mathbf{D}) = \mathbf{D} - \left(\frac{1}{2\nu}\left|\frac{\mathbf{S}}{2\nu}\right|^{r'-2} + \frac{\tau_y}{\sqrt{\varepsilon^2 + |\mathbf{S}|^2}}\right)\mathbf{S},$$

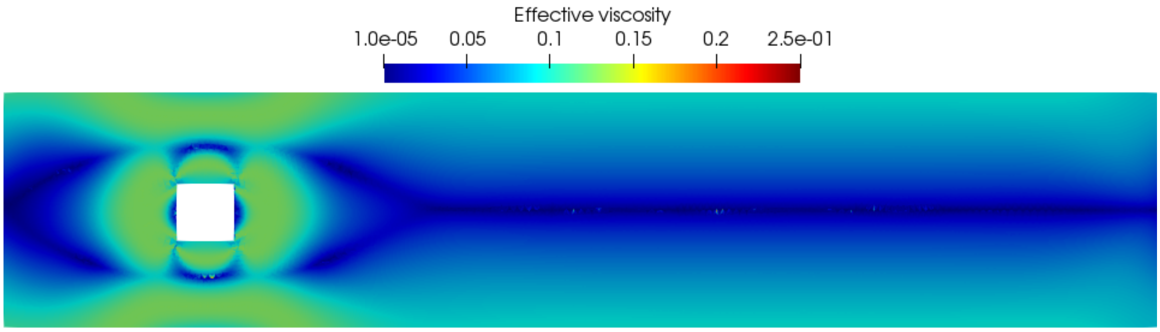
γ	k	# refs	# dofs	Γ_2 10	1000	5000	10000
10^4	2	1	3.1×10^4	4	3.66	4	4
		2	1.2×10^5	3.66	3.66	4	3.5
		3	4.9×10^5	3.66	3.66	4	4
	3	1	6.5×10^4	2.33	2	2.5	2.5
		2	2.5×10^5	2.33	2	2.5	2.5
		3	1.0×10^6	2.33	2	2.5	2.5
1	2	1	3.1×10^4	9.66	20.3	32	34.5
		2	1.2×10^5	9	19.3	30.5	31.5
		3	4.9×10^5	8	17.3	26	27
	3	1	6.5×10^4	7.33	15.6	24	26
		2	2.5×10^5	6.33	13.6	20	21
		3	1.0×10^6	6	11.3	16.5	17.5

Table 4.6: Average number of Krylov iterations per Newton step as Γ_2 increases for the 2D generalised Carreau–Yasuda relation with $\nu = 0.01$, $r_1 = 1.7$, $r_2 = 3$, $\Gamma_1 = 10$, $\beta_1 = 0.2$, $\beta_2 = 0.9$.

where $\varepsilon > 0$. The problem was solved on the set $\Omega = (0,2) \times (0,0.41) \setminus (0.3,0.4) \times (0.15,0.25)$, with boundary data

$$\begin{cases} \boldsymbol{u} = (4\frac{0.3x_2(0.41-x_2)}{0.41^2}, 0)^\top, & \text{on } \partial\Omega_1 := \{x_1 = 0\} \cap \partial\Omega, \\ \boldsymbol{u}_\tau = 0 \text{ and } \mathbf{S}\boldsymbol{n} \cdot \boldsymbol{n} - p = 0 & \text{on } \partial\Omega_2 := \{x_1 = 2\} \cap \partial\Omega, \\ \boldsymbol{u} = \boldsymbol{0}, & \text{on } \partial\Omega \setminus (\partial\Omega_1 \cup \partial\Omega_2), \end{cases} \tag{4.46}$$

where $\boldsymbol{n}$ is the outward normal vector to the boundary and $\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}$ is the tangential part of the velocity. Table 4.7 shows the number of Krylov iterations per Newton step obtained using two full multigrid cycles with 3 relaxation steps per level as $\tilde{A}^{-1}$; the same robust behaviour as in the previous examples can be observed here. Figure 4.8 shows the effective viscosity $\mu_{\text{eff}} := \frac{\mathbf{S}}{2\mathbf{D}}$ for the solution of this problem and for that of a regular shear-thinning power-law fluid. It can be observed that the effective viscosity of the activated fluid greatly decreases far away from the obstacle, which is a common assumption in the study of boundary layers.



(a) Effective viscosity for an activated Euler/power-law fluid with $r = 1.3$, $\tau_y = 3$, $\nu = 0.5$, and $\varepsilon = 1 \times 10^{-5}$.

(b) Effective viscosity for a power-law fluid with $r = 1.3$ and $\nu = 0.5$.

Figure 4.8: Effective viscosity for the flow past an obstacle.

k	# refs	# dofs	ε			
			0.2	0.01	0.0001	0.00001
2	1	3.5×10^4	5	3	2	2
	2	1.4×10^5	5.66	4	2	2
	3	5.6×10^5	4.6	4	3	3
3	1	7.3×10^4	2.66	2	1	1
	2	2.9×10^5	3	2	2	2
	3	1.1×10^6	3	2	2	2

Table 4.7: Average number of Krylov iterations per Newton step as ε decreases for the Euler/power-law relation with $\nu = 0.5$, $r = 1.3$, $\tau_y = 3$.

Chapter 5

The Anisothermal Problem

As mentioned in Chapter 1, in applications the thermal effects are often of fundamental importance and cannot be neglected. The purpose of this chapter is to extend the results from previous chapters to cover the numerical approximation of heat-conducting flow.

5.1 Unsteady forced convection flow

In this section we will focus on extending the convergence result presented in Chapter 3 to the non-isothermal system

$$
\begin{aligned}
\partial_t \boldsymbol{u} - \operatorname{div}(\mathbf{S} - \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p &= \boldsymbol{f} && \text{in } (0,T) \times \Omega, \\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } (0,T) \times \Omega, \\
\partial_t E + \operatorname{div}((E+p)\boldsymbol{u} - \mathbf{S}\boldsymbol{u}) - \operatorname{div}(\kappa(\theta)\nabla\theta) &= \boldsymbol{f} \cdot \boldsymbol{u} && \text{in } (0,T) \times \Omega, \\
(\mathbf{D}(\boldsymbol{u}), \mathbf{S}, \theta) &\in \mathcal{A}(\cdot) && \text{a.e. in } (0,T) \times \Omega, \\
\boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } (0,T) \times \partial\Omega, \\
\kappa(\theta)\nabla\theta \cdot \boldsymbol{n} &= 0 && \text{on } (0,T) \times \partial\Omega, \\
\alpha \boldsymbol{u}_\tau + (\mathbf{S}\boldsymbol{n})_\tau &= 0 && \text{on } (0,T) \times \partial\Omega, \\
\boldsymbol{u}(0,\cdot) &= \boldsymbol{u}_0(\cdot) && \text{in } \Omega, \\
\theta(0,\cdot) &= \theta_0(\cdot) && \text{in } \Omega,
\end{aligned}
\tag{5.1}
$$

where $\alpha > 0$ and $\kappa, \boldsymbol{f}, \boldsymbol{u}_0, \theta_0$ are given functions. Here $\boldsymbol{n}$ is the outward unit normal vector, θ is the temperature, and E is the sum of internal and kinetic energies:

$$
E = \frac{1}{2}|\boldsymbol{u}|^2 + c_v \theta, \tag{5.2}
$$

where $c_v > 0$ is the specific heat capacity at constant volume. The tangential part $\boldsymbol{v}_\tau$ of a vector $\boldsymbol{v}$ is defined as $\boldsymbol{v}_\tau := \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$. The graph $\mathcal{A}(\cdot) \subset \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}$ now includes the temperature and could be thought of as a regular graph for the stress and symmetric velocity gradient, but now parametrised by the temperature. The existence of weak solutions to the problem (5.1) was shown in [MZ18] (see also [BFM09]).

By multiplying the momentum equation by $\boldsymbol{u}$ and substracting the resulting equation from the balance of total energy, we find the form of the temperature equation most commonly found in applications:

$$\partial_t \theta + \operatorname{div}(\theta \boldsymbol{u} - \kappa(\theta)\nabla\theta) = \mathbf{S} : \mathbf{D}(\boldsymbol{u}) \quad \text{in } (0,T)\times\Omega. \tag{5.3}$$

Equation (5.3) has the advantage that it does not contain the pressure and so introduces fewer couplings between the different unknowns. The major difficulty is that it is then not clear how to pass to the limit in the term $\mathbf{S} : \mathbf{D}(\boldsymbol{u})$, which a priori only belongs to $L^1(Q)$. The two forms of the energy equation are equivalent whenever testing with the velocity $\boldsymbol{u}$ is allowed; the key insight from Bulíček, Málek and Feireisl in [BFM09], is that the equation for the total energy contained in (5.1) is more amenable to weak convergence arguments, and is therefore preferable in the analysis. Whenever the velocity is not an admissible test function (e.g. as in the case of the 3D Navier–Stokes system), the relation (5.3) can be obtained in the form of an inequality, which plays the role of an entropy inequality:

$$\partial_t \theta + \operatorname{div}(\theta \boldsymbol{u} - \kappa(\theta)\nabla\theta) \geq \mathbf{S} : \mathbf{D}(\boldsymbol{u}) \quad \text{in } (0,T)\times\Omega. \tag{5.4}$$

In this chapter we will work with an implicit constitutive relation defined by the graph

$$(\mathbf{D},\mathbf{S},\theta) \in \mathcal{A}(\cdot) \iff 2\mu(\theta)\frac{(|\mathbf{D}| - \sigma(\theta))^+}{|\mathbf{D}|}\mathbf{D} = \frac{(|\mathbf{S}| - \tau(\theta))^+}{|\mathbf{S}|}\mathbf{S}, \tag{5.5}$$

where $\tau, \sigma, \mu \colon \mathbb{R} \to \mathbb{R}$ are continuous functions such that

$$\begin{gathered} 0 \leq \tau(s), \sigma(s) \leq c_0, \\ c_1 \leq \mu(s) \leq c_2, \\ \tau(s)\sigma(s) = 0, \end{gathered} \tag{5.6}$$

for all $s \in \mathbb{R}$, for some positive constants c_0, c_1, c_2. We will also assume that the heat conductivity κ is a continuous function such that $c_1 \leq \kappa(s) \leq c_2$, for any $s \in \mathbb{R}$. The graph (5.5) defines a fluid with either Bingham or activated Euler rheology in which the viscosity and activation parameters may depend on the temperature. Naturally, this family of constitutive relations also includes the Navier–Stokes model with a temperature-dependent viscosity (when $\tau \equiv 0 \equiv \sigma$). The graph (5.5) was introduced in [MZ18], where it was proved that, assuming the conditions (5.6) hold, it then satisfies monotonicity and coercivity conditions analogous to (A2) and (A4) from the isothermal case.

Lemma 5.1.1 ([MZ18, Lemma 3]). *Consider the graph $\mathcal{A} \subset \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}$ defined by (5.5). If the conditions stated in (5.6) hold, then there exist two positive numbers $\beta_1, \beta_2 > 0$ such that*

$$\mathbf{S} : \mathbf{D} \geq \beta_1(|\mathbf{S}|^2 + |\mathbf{D}|^2) - \beta_2, \tag{5.7a}$$

$$(\mathbf{S} - \overline{\mathbf{S}}) : (\mathbf{D} - \overline{\mathbf{D}}) \geq 0, \tag{5.7b}$$

for any $(\mathbf{S}, \mathbf{D}, \theta), (\overline{\mathbf{S}}, \overline{\mathbf{D}}, \theta) \in \mathcal{A}$.

Proof. This proof was presented in [MZ18], but we reproduce it here for the reader's convenience. Suppose that $\sigma \equiv 0$; the case $\tau \equiv 0$ is treated analogously. Then, for an arbitrary $(\mathbf{S}, \mathbf{D}, \theta) \in \mathcal{A}$ we can write

$$\mathbf{D} = \frac{1}{2\mu(\theta)} \frac{(\mathbf{S} - \tau(s))^+}{|\mathbf{S}|} \mathbf{S}.$$

From (5.6) we then obtain

$$\begin{aligned} \mathbf{S} : \mathbf{D} &= \frac{|\mathbf{S}|}{2\mu(\theta)}(|\mathbf{S}| - \tau(\theta))^+ \geq \frac{1}{2\mu(\theta)}((|\mathbf{S}| - \tau(\theta))^+)^2 \\ &\geq \frac{1}{2c_2}((|\mathbf{S}| - c_0)^+)^2 \geq \frac{1}{2c_2}\left(\frac{|\mathbf{S}|^2}{4} - c_0^2\right). \end{aligned} \tag{5.8}$$

On the other hand we have that

$$|\mathbf{D}|^2 = \frac{1}{(2\mu(\theta))^2}((|\mathbf{S}| - \tau(\theta))^+)^2,$$

which implies that

$$\mathbf{S} : \mathbf{D} \geq 2\mu(\theta)|\mathbf{D}|^2 \geq 2c_1|\mathbf{D}|^2. \tag{5.9}$$

Combining (5.8) and (5.9) yields (5.7a). Now, observing that $\mu > 0$ and that the mapping

$$\boldsymbol{\sigma} \mapsto \frac{(|\boldsymbol{\sigma}| - \tau(\theta))^+}{|\boldsymbol{\sigma}|}\boldsymbol{\sigma},$$

is monotone, for $\theta \in \mathbb{R}$ fixed, we conclude that for arbitrary $(\mathbf{S}, \mathbf{D}, \theta), (\overline{\mathbf{S}}, \overline{\mathbf{D}}, \theta) \in \mathcal{A}$:

$$(\mathbf{S} - \overline{\mathbf{S}}) : (\mathbf{D} - \overline{\mathbf{D}}) = \frac{1}{2\mu(\theta)}(\mathbf{S} - \overline{\mathbf{S}}) : \left(\frac{(|\mathbf{S}| - \tau(\theta))^+}{\mathbf{S}}\mathbf{S} - \frac{(|\overline{\mathbf{S}}| - \tau(\theta))^+}{\overline{\mathbf{S}}}\overline{\mathbf{S}}\right) \geq 0,$$

which proves (5.7b). $\square$

In the same spirit as [MZ18], in the numerical scheme we will employ a sequence of continuous explicit approximations of the implicit constitutive relation (5.5); we will say that $(\mathbf{D}, \mathbf{S}, \theta) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}$ belongs to the approximate graph $\mathcal{A}^k$, for $k \in \mathbb{N}$, if either of the two following relations hold:

$$\begin{aligned} \mathbf{D} = \boldsymbol{\mathcal{D}}^k(\mathbf{S},\theta) &:= \min\left\{ n + \frac{1}{2\mu(\theta)}, \frac{\frac{1}{2\mu(\theta)}(|\mathbf{S}| - \tau(\theta))^+ + \sigma(\theta)}{|\mathbf{S}|} \right\}, \\ \mathbf{S} = \boldsymbol{\mathcal{S}}^k(\mathbf{D},\theta) &:= \min\left\{ n + 2\mu(\theta), \frac{2\mu(\theta)(|\mathbf{D}| - \sigma(\theta))^+ + \tau(\theta)}{|\mathbf{D}|} \right\}. \end{aligned} \tag{5.10}$$

Either of the two can be chosen, depending on whether one wishes to consider explicit approximations of the stress in terms of the symmetric velocity gradient or vice-versa. The functions $\boldsymbol{\mathcal{D}}^k$ and $\boldsymbol{\mathcal{S}}^k$ satisfy, uniformly in $k \in \mathbb{N}$, the same coercivity and monotonicity properties as those stated in Lemma 5.1.1. More importantly, there is also a localised Minty lemma available.

Lemma 5.1.2 ([MZ18, Lemma 6])**.** *Let $U \subset Q$ be measurable, let $\mathcal{A}$ be defined by (5.5) and let $\mathcal{A}^k$ be defined by (5.10). Now suppose $\{\mathbf{S}^k\}_{k\in\mathbb{N}}$, $\{\mathbf{D}^k\}_{k\in\mathbb{N}}$ and $\{\theta^k\}_{k\in\mathbb{N}}$ are sequences of measurable functions on Q satisfying*

$$\begin{aligned} (\mathbf{D}^k, \mathbf{S}^k, \theta^k) &\in \mathcal{A}^k && \text{a.e. in } U, \\ \mathbf{S}^k &\rightharpoonup \mathbf{S} && \text{weakly in } L^2(U)^{d\times d}, \\ \mathbf{D}^k &\rightharpoonup \mathbf{D} && \text{weakly in } L^2(U)^{d\times d}, \\ \theta^k &\to \theta && \text{a.e. in } U, \\ \limsup_{k\to\infty} \int_U \mathbf{S}^k : \mathbf{D}^k &\leq \int_U \mathbf{S} : \mathbf{D}. \end{aligned}$$

Then $(\mathbf{D}, \mathbf{S}, \theta) \in \mathcal{A}$ almost everywhere in U and $\mathbf{S}^k : \mathbf{D}^k \rightharpoonup \mathbf{S} : \mathbf{D}$ weakly in $L^1(U)$.

An advantage of Lemma 5.1.2 is that the sequence is only required to belong to the approximate graph $\mathcal{A}^k$, which allows one to take the graph approximation limit concurrently with other limits, as opposed to a graph approximation based on mollification (c.f. Chapter 3). The same conclusion holds in the lemma assuming the stronger condition $(\mathbf{D}^k, \mathbf{S}^k, \theta^k) \in \mathcal{A}$.

5.1.1 Convergence of the finite element approximations

Let us define

$$\check{2} := \min\left\{ \frac{d+2}{d}, 2 \right\}. \tag{5.11}$$

Now, suppose we are given an initial velocity field $\boldsymbol{u}_0 \in L^2_{\mathrm{div}}(\Omega)^d$, an initial temperature distribution $\theta_0 \in L^1(\Omega)$ satisfying $\theta_0 \geq \overline{c}$ for some positive constant $\overline{c}$, a body force $\boldsymbol{f} \in L^2(0,T;W^{-1,2}_{\boldsymbol{n}}(\Omega)^d)$, and a graph $\mathcal{A}$ satisfying (5.5) and (5.6). The weak formulation for (5.1) then reads as follows.

Formulation $\tilde{\mathbf{A}}$. Find functions

$$
\begin{gathered}
\mathbf{S} \in L^2_{\mathrm{sym,tr}}(Q)^{d\times d},\\
\boldsymbol{u} \in L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}(\Omega)^d) \cap L^\infty(0,T;L^2_{\mathrm{div}}(\Omega)^d),\\
\partial_t \boldsymbol{u} \in L^{\check{2}}(0,T;W^{-1,\check{2}}_{\boldsymbol{n}}(\Omega)^d),\\
p \in L^{\check{2}}_0(Q),\\
\theta \in L^\infty(0,T;L^1(\Omega)) \cap L^n(0,T;W^{1,n}(\Omega)) \text{ for } n \in [1,\tfrac{5}{4}),\\
\partial_t\theta \in \mathcal{M}(0,T;W^{1,m'}(\Omega)^*) \quad \text{for } m \in [1,\tfrac{10}{9}),\\
\partial_t E \in L^1(0,T;W^{1,m'}(\Omega)^*) \quad \text{for } m \in [1,\tfrac{10}{9}),
\end{gathered}
$$

where we are denoting the total energy by $E := \frac{|\boldsymbol{u}|^2}{2} + \theta$, such that:

$$
\begin{aligned}
&\langle \partial_t \boldsymbol{u}, \boldsymbol{v}\rangle + \int_\Omega (\mathbf{S} - \boldsymbol{u}\otimes\boldsymbol{u} - p\mathbf{I}) : \mathbf{D}(\boldsymbol{v})\\
&\qquad\qquad + \alpha\int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{v} = \langle \boldsymbol{f},\boldsymbol{v}\rangle \qquad \forall\, \boldsymbol{v} \in W^{1,\check{2}'}_{\boldsymbol{n}}(\Omega)^d, \text{ a.e. } t\in(0,T),\\
&\langle \partial_t E, \psi\rangle + \int_\Omega (\mathbf{S}\boldsymbol{u} - (E+p)\boldsymbol{u} + \kappa(\theta)\nabla\theta)\cdot\nabla\psi\\
&\qquad\qquad + \alpha\int_{\partial\Omega} |\boldsymbol{u}|^2\psi = \langle \boldsymbol{f}, \boldsymbol{u}\psi\rangle \qquad \forall\,\psi \in W^{1,\infty}(\Omega), \text{ a.e. } t \in (0,T),
\end{aligned}
$$

$$
\begin{gathered}
(\mathbf{D}(\boldsymbol{u}), \mathbf{S}, \theta) \in \mathcal{A}, \text{ a.e. in } Q,\\
\theta \geq \overline{c} \quad \text{a.e. in } Q,\\
\operatorname*{ess\,lim}_{t\to 0^+} \|\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_0(\cdot)\|_{L^2(\Omega)} = 0,\\
\operatorname*{ess\,lim}_{t\to 0^+} \|\theta(t,\cdot) - \theta_0(\cdot)\|_{L^1(\Omega)} = 0.
\end{gathered}
$$

In addition, we have that the following entropy inequality holds in the sense of measures

$$
\langle \partial_t\theta, \psi\rangle + \int_\Omega (-\theta\boldsymbol{u} + \kappa(\theta)\nabla\theta)\cdot\nabla\boldsymbol{u} \geq \int_\Omega \mathbf{S} : \mathbf{D}(\boldsymbol{u})\psi \quad \forall\,\psi \in W^{1,\infty}(\Omega), \psi \geq 0,
$$

with equality achieved if $d = 2$. The ranges for n and m are written for $d = 3$, which is the more restrictive case.

In this section we will assume that the space of discrete pressures M^n is $H^1(\Omega)$-conforming; we could for instance employ the Taylor–Hood or the MINI element for the velocity and pressure. The result still holds for non-conforming elements, assuming one makes sure the approximation of a Laplace problem using the finite element space M^n is well posed (e.g. by employing an interior penalty formulation of the Laplacian when using piecewise discontinuous pressures), but this generalisation would only introduce technical complications without providing additional insight and we therefore avoid it to improve readability. The motivation behind this assumption is that we will employ a quasi-compressibility approximation of the form

$$\operatorname{div} \boldsymbol{u} = \varepsilon \Delta p,$$

where eventually the limit $\varepsilon \to 0$ will be taken. This will ensure that uniform estimates for the pressure that make it possible to pass to the limit are available.

For $j \in \{1, \ldots, T/\tau_m\}$ let us denote the time averages of $\boldsymbol{f}$ by $\boldsymbol{f}_j \in W_{\boldsymbol{n}}^{-1,2}(\Omega)^d$ (see (2.26)). Similarly to Chapter 3, the problem will be discretised in time using backward differences.

In order to restore admissibility of the velocity we will work with the modified convective term defined, for $k \in \mathbb{N}$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in C_{\boldsymbol{n}}^{\infty}(\Omega)^d$, by

$$\mathcal{B}_k(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := \begin{cases} -\displaystyle\int_\Omega \boldsymbol{v} \otimes ((\phi_k \boldsymbol{u}) * \omega_{1/k})_{\mathrm{div}} : \nabla \boldsymbol{w}, & \text{if } d = 3, \\ \dfrac{1}{2} \displaystyle\int_\Omega \boldsymbol{w} \otimes \boldsymbol{u} : \nabla \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{u} : \nabla \boldsymbol{w}, & \text{if } d = 2, \end{cases}$$

where $\omega_{1/k}$ is a standard mollifier over a ball of radius $1/k$, ϕ_k is a smooth function such that $\operatorname{dist}(\operatorname{supp}\phi_k, \partial\Omega) \geq 1/k$, and $\boldsymbol{f}_{\mathrm{div}}$ represents the solenoidal part of a function $\boldsymbol{f}$ (recall (2.4)). Note that the continuity properties of the Helmholtz decomposition imply that if $\boldsymbol{v}^n \to \boldsymbol{v}$ strongly in $L^s(\Omega)^d$ for some $s \in [1, \infty)$ as $n \to \infty$, then

$$((\phi_k \boldsymbol{v}^n) * \omega_{1/k})_{\mathrm{div}} \to ((\phi_k \boldsymbol{v}) * \omega_{1/k})_{\mathrm{div}} \quad \text{strongly in } L^s(\Omega)^d, \tag{5.12}$$

and moreover if $\boldsymbol{v}^k \to \boldsymbol{v}$ strongly in $L^s(\Omega)^d$ as $k \to \infty$ for some $s \in [1, , \infty)$ and $\operatorname{div} \boldsymbol{v} = 0$, then

$$((\phi_k \boldsymbol{v}^k) * \omega_{1/k})_{\mathrm{div}} \to \boldsymbol{v} \quad \text{strongly in } L^s(\Omega)^d. \tag{5.13}$$

The modified convective term defined above also has the usual skew-symmetry property

$$\mathcal{B}_k(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in C_{\boldsymbol{n}}^{\infty}(\Omega)^d, \tag{5.14}$$

regardless of whether $\boldsymbol{u}$ is actually divergence-free or not, which is useful when obtaining a priori estimates. In the energy equation we will employ the convective term defined for $\theta, \psi \in C^\infty(\Omega)$ and $\boldsymbol{u} \in C^\infty_{\boldsymbol{n}}(\Omega)^d$:

$$\mathcal{C}_k(\boldsymbol{u}, \theta, \psi) := -\int_\Omega \theta((\phi_k \boldsymbol{u}) * \omega_{1/k})_{\mathrm{div}} \cdot \nabla\psi. \tag{5.15}$$

This trilinear form also has the property that

$$\mathcal{C}_k(\boldsymbol{u}, \theta, \theta) = 0 \qquad \forall\, \boldsymbol{u} \in C^\infty_{\boldsymbol{n}}(\Omega)^d, \theta \in C^\infty(\Omega). \tag{5.16}$$

Formulation $\tilde{\mathbf{A}}_{\mathbf{k,n,m,l}}$. Find functions $\mathbf{S}_j^{k,n,m,l} \in \Sigma^n$, $\boldsymbol{u}_j^{k,n,m,l} \in V^n$, $p_j^{k,n,m,l} \in M_0^n$, and $\theta_j^{k,n,m,l} \in U^m$, for $j \in \{1, \ldots, T/\tau_m\}$, such that:

$$\begin{aligned}
&\int_\Omega (\boldsymbol{\mathcal{D}}^k(\mathbf{S}_j^{k,n,m,l}, \theta_j^{k,n,m,l}) - \mathbf{D}(\boldsymbol{u}_j^{k,n,m,l})) : \boldsymbol{\tau} = 0 && \forall\, \boldsymbol{\tau} \in \Sigma^n,\\
&\frac{1}{\tau_m}\int_\Omega (\boldsymbol{u}_j^{k,n,m,l} - \boldsymbol{u}_{j-1}^{k,n,m,l}) \cdot \boldsymbol{v} + \alpha \int_{\partial\Omega} \boldsymbol{u}_j^{k,n,m,l} \cdot \boldsymbol{v} - \int_\Omega p_j^{k,n,m,l} \operatorname{div} \boldsymbol{v} &&\\
&\qquad + \int_\Omega \mathbf{S}_j^{k,n,m,l} : \mathbf{D}(\boldsymbol{v}) + \mathcal{B}_k(\boldsymbol{u}_j^{k,n,m,l}, \boldsymbol{u}_j^{k,n,m,l}, \boldsymbol{v}) = \langle \boldsymbol{f}_j, \boldsymbol{v} \rangle && \forall\, \boldsymbol{v} \in V^n,\\
&-\int_\Omega q \operatorname{div} \boldsymbol{u}_j^{k,n,m,l} = \frac{1}{l}\int_\Omega \nabla p_j^{k,n,m,l} \cdot \nabla q && \forall\, q \in M^n,\\
&\frac{1}{\tau_m}\int_\Omega (\theta_j^{k,n,m,l} - \theta_{j-1}^{k,n,m,l})\psi + \int_\Omega \kappa(\theta_j^{k,n,m,l}) \nabla\theta_j^{k,n,m,l} \cdot \nabla\psi &&\\
&\qquad + \mathcal{C}_k(\boldsymbol{u}_j^{k,n,m,l}, \theta_j^{k,n,m,l}, \psi) = \int_\Omega \mathbf{S}_j^{k,n,m,l} : \boldsymbol{\mathcal{D}}^k(\mathbf{S}_j^{k,n,m,l}, \theta_j^{k,n,m,l})\psi && \forall\, \psi \in U^m,\\
&\boldsymbol{u}_0^{k,n,m,l} = P_V^n \boldsymbol{u}_0, &&\\
&\theta_0^{k,n,m,l} = P_U^m \theta_0^n, &&
\end{aligned}$$

where P_V^n and P_U^m denote the L^2-projections onto V^n and U^m, respectively, and $\theta_0^n := (\omega_{1/n} * \theta_0)$, where $\omega_{1/n}$ is a mollification kernel of radius $1/n$ (it is understood that $\theta_0 = \overline{c}$ on $\mathbb{R}^d \setminus \Omega$). With this we then have that $\theta_0^n \geq \overline{c}$ almost everywhere, and that

$$\begin{aligned}
\boldsymbol{u}_0^{k,n,m,l} &\to \boldsymbol{u}_0 && \text{strongly in } L^2(\Omega)^d, \text{ as } n \to \infty,\\
\theta_0^{k,n,m,l} &\to \theta_0^n && \text{strongly in } L^2(\Omega)^d, \text{ as } m \to \infty,\\
\theta_0^n &\to \theta_0 && \text{strongly in } L^1(\Omega)^d, \text{ as } n \to \infty.
\end{aligned} \tag{5.17}$$

In Formulation $\tilde{\mathrm{A}}_{\mathrm{k,n,m,l}}$, the indices k, m, n, and l, refer to the approximation of the implicit graph and convection term, the time discretisation and Galerkin limit for the temperature, the Galerkin limit for the rest of the unknowns, and the quasi-compressibility approximation, respectively. As in the proof of Theorem 3.1.5, in the convergence argument we will need to take these limits in succession.

Lemma 5.1.3. *Suppose the material parameters satisfy condition* (5.6) *and suppose that* $\{\Sigma^n, U^n, V^n, M^n\}_{n\in\mathbb{N}}$ *is a family of finite element spaces satisfying Assumptions 2.5.2–2.5.5. Then, for the solutions of Formulation* $\tilde{\mathrm{A}}_{\mathrm{k,n,m,l}}$ *the following a priori estimate holds:*

$$\sup_{j\in\{1,\dots,T/\tau_m\}} \|\boldsymbol{u}_j^{k,n,m,l}\|^2_{L^2(\Omega)} + \sum_{j=1}^{T/\tau_m} \|\boldsymbol{u}_j^{k,n,m,l} - \boldsymbol{u}_{j-1}^{k,n,m,l}\|^2_{L^2(\Omega)} + \tau_m \sum_{j=1}^{T/\tau_m} \|\mathbf{S}_j^{k,n,m,l}\|^2_{L^2(\Omega)}$$
$$+ \tau_m \sum_{j=1}^{T/\tau_m} \|\mathbf{D}(\boldsymbol{u}_j^{k,n,m,l})\|^2_{L^2(\Omega)} + \tau_m \sum_{j=1}^{T/\tau_m} \alpha \|\boldsymbol{u}_j^{k,n,m,l}\|^2_{L^2(\partial\Omega)} + \frac{\tau_m}{l} \sum_{j=1}^{T/\tau_m} \|\nabla p_j^{k,n,m,l}\|^2_{L^2(\Omega)}$$
$$+ \tau_m \sum_{j=1}^{T/\tau_m} \|\boldsymbol{\mathcal{D}}^k(\mathbf{S}_j^{k,n,m,l}, \theta_j^{k,n,m,l})\|^2_{L^2(\Omega)} \leq c,$$

$$\sup_{j\in\{1,\dots,T/\tau_m\}} \|\theta_j^{k,n,m,l}\|^2_{L^2(\Omega)} + \sum_{j=1}^{T/\tau_m} \|\theta_j^{k,n,m,l} - \theta_{j-1}^{k,n,m,l}\|^2_{L^2(\Omega)} + \tau_m \sum_{j=1}^{T/\tau_m} \|\nabla \theta_j^{k,n,m,l}\|^2_{L^2(\Omega)} \leq c_n,$$

where $c>0$ *is independent of* k,m,n,l *but* $c_n > 0$ *might depend on* n.

Proof. The first estimate can be obtained by testing the constitutive relation and momentum and mass equations with $(\mathbf{S}_j^{k,n,m,l}, \boldsymbol{u}_j^{k,n,m,l}, p_j^{k,n,m,l})$ and the proof is analogous to that of the isothermal case; the inf-sup condition (2.23) is again needed to control the norm $\|\mathbf{D}(\boldsymbol{u}_j^{k,n,m,l})\|_{L^2(\Omega)}$. For the second estimate, by testing the discrete energy equation with $\theta_j^{k,n,m,l}$ we obtain:

$$\begin{aligned}
&\frac{1}{2\tau_m}(\|\theta_j^{k,n,m,l}\|^2_{L^2(\Omega)} - \|\theta_{j-1}^{k,n,m,l}\|^2_{L^2(\Omega)} + \|\theta_j^{k,n,m,l} - \theta_{j-1}^{k,n,m,l}\|^2_{L^2(\Omega)}) + c_1 \|\nabla \theta_j^{k,n,m,l}\|^2_{L^2(\Omega)} \\
&\qquad \leq \int_\Omega \mathbf{S}_j^{k,n,m,l} : \boldsymbol{\mathcal{D}}^k(\mathbf{S}_j^{k,n,m,l}, \theta_j^{k,n,m,l}) \theta_j^{k,n,m,l} \\
&\qquad \leq \|\mathbf{S}_j^{k,n,m,l}\|_{L^\infty(\Omega)} \|\boldsymbol{\mathcal{D}}^k(\mathbf{S}_j^{k,n,m,l}, \theta_j^{k,n,m,l})\|_{L^2(\Omega)} \|\theta_j^{k,n,m,l}\|_{L^2(\Omega)} \\
&\qquad \leq \tilde{c}_n \|\mathbf{S}_j^{k,n,m,l}\|_{L^2(\Omega)} \|\boldsymbol{\mathcal{D}}^k(\mathbf{S}_j^{k,n,m,l}, \theta_j^{k,n,m,l})\|_{L^2(\Omega)} \|\theta_j^{k,n,m,l}\|_{L^2(\Omega)},
\end{aligned}$$

where $\tilde{c}_n > 0$ is a norm-equivalence constant that might blow up with n. Therefore, we have for an arbitrary $i \in \{1,\dots,T/\tau_m\}$:

$$\|\theta_i^{k,n,m,l}\|^2_{L^2(\Omega)} + \sum_{j=1}^{i} \|\theta_j^{k,n,m,l} - \theta_{j-1}^{k,n,m,l}\|^2_{L^2(\Omega)} + \tau_m \sum_{j=1}^{i} \|\nabla \theta_j^{k,n,m,l}\|^2_{L^2(\Omega)}$$
$$\leq \tilde{c}_n \sum_{j=1}^{i} \tau_m \|\mathbf{S}_j^{k,n,m,l}\|_{L^2(\Omega)} \|\boldsymbol{\mathcal{D}}^k(\mathbf{S}_j^{k,n,m,l}, \theta_j^{k,n,m,l})\|_{L^2(\Omega)} \|\theta_j^{k,n,m,l}\|_{L^2(\Omega)} + \|\theta_0^{k,n,m,l}\|^2_{L^2(\Omega)}$$
$$\leq \tilde{c}_n \sum_{j=1}^{i} [(\tau_m \|\mathbf{S}_j^{k,n,m,l}\|^2_{L^2(\Omega)})^{1/2} (\tau_m \|\boldsymbol{\mathcal{D}}^k(\mathbf{S}_j^{k,n,m,l}, \theta_j^{k,n,m,l})\|^2_{L^2(\Omega)})^{1/2}]^2$$

$$
\begin{aligned}
&+\sum_{j=1}^{i}\|\theta_j^{k,n,m,l}\|^2_{L^2(\Omega)}+\|\theta_0^{k,n,m,l}\|^2_{L^2(\Omega)}\\
&\leq c_n+\sum_{j=1}^{i}\|\theta_j^{k,n,m,l}\|^2_{L^2(\Omega)},
\end{aligned}
$$

where we used the first estimate and the fact that $\|\theta_0^{k,n,m,l}\|_{L^2(\Omega)} \leq \|\theta_0^n\|_{L^2(\Omega)}$. Noting that without any loss of generality we can assume that $\tau_m \leq 1$, the result follows by a discrete version of Gronwall's inequality [Rou13, Eq. (1.69)].

□

Observe that Lemma 5.1.3, combined with the Korn and Poincaré inequalities (2.2), implies that we also have the uniform estimate

$$
\tau_m \sum_{j=1}^{T/\tau_m} \|\boldsymbol{u}_j^{k,n,m,l}\|^2_{W^{1,2}(\Omega)} \leq c. \tag{5.19}
$$

The following lemma now guarantees the existence of discrete solutions.

Lemma 5.1.4. *Suppose that the same assumptions as in Lemma 5.1.3 hold. Then, for any $j \in \{1,\dots,T/\tau_m\}$, given $(\boldsymbol{u}_{j-1}^{k,n,m,l},\theta_{j-1}^{k,n,m,l}) \in V^n_{\mathrm{div}} \times U^n$, there exists a tuple $(\mathbf{S}_j^{k,n,m,l},\boldsymbol{u}_j^{k,n,m,l},p_j^{k,n,m,l},\theta_j^{k,n,m,l}) \in \Sigma^n \times V^n \times M_0^n \times U^n$ that solves Formulation $\tilde{\mathrm{A}}_{\mathrm{k,n,m,l}}$.*

Proof. The proof will make use of a fixed point argument. Let the initial guess be $(\boldsymbol{u}^0,\theta^0) := (\boldsymbol{u}_{j-1}^{k,n,m,l},\theta_{j-1}^{k,n,m,l})$, and then define iteratively $(\mathbf{S}^i,\boldsymbol{u}^i,p^i) \in \Sigma^n \times V^n \times M_0^n$, for $i \in \mathbb{N}$, as the solution of

$$
\begin{aligned}
\int_\Omega (\boldsymbol{\mathcal{D}}^k(\mathbf{S}^i,\theta^{i-1}) - \mathbf{D}(\boldsymbol{u}^i)) : \boldsymbol{\tau} = 0 \qquad & \forall\, \boldsymbol{\tau} \in \Sigma^n,\\
\frac{1}{\tau_m}\int_\Omega (\boldsymbol{u}^i - \boldsymbol{u}^{i-1})\cdot\boldsymbol{v} + \alpha\int_{\partial\Omega}\boldsymbol{u}^i\cdot\boldsymbol{v} + \int_\Omega \mathbf{S}^i : \mathbf{D}(\boldsymbol{v}) + \mathcal{B}_k(\boldsymbol{u}^i,\boldsymbol{u}^i,\boldsymbol{v})&\\
-\int_\Omega p^i\,\mathrm{div}\boldsymbol{v} = \langle \boldsymbol{f}_j,\boldsymbol{v}\rangle \qquad & \forall\, \boldsymbol{v} \in V^n,\\
-\int_\Omega \mathrm{div}\,\boldsymbol{u}^i q = \frac{1}{l}\int_\Omega \nabla p^i\cdot\nabla q \qquad & \forall\, q \in M^n,
\end{aligned}
$$

and then let $\theta^i \in U^m$ be the solution of

$$
\frac{1}{\tau_m}\int_\Omega (\theta^i - \theta^{i-1})\psi + \int_\Omega \kappa(\theta^i)\nabla\theta^i\cdot\nabla\psi + \mathcal{C}_k(\theta^i,\boldsymbol{u}^i,\psi) = \int_\Omega \mathbf{S}^i : \boldsymbol{\mathcal{D}}^k(\mathbf{S}^i,\theta^{i-1})\psi \quad \forall\, \psi \in U^m.
$$

A combination of analogous estimates to the ones described in Lemma 5.1.3 and a corollary of Brouwer's fixed point theorem [GR86, Ch. 4 Cor. 1.1] guarantee the existence of solutions for every $i \in \mathbb{N}$. These same estimates and a compactness argument conclude the proof of the lemma. In this case the passage to the limit in i is straightforward since the spaces are finite-dimensional and the convergences are therefore strong. $\square$

In terms of the time-interpolants, the discrete formulation can be written in the form

$$
\begin{aligned}
&\int_\Omega (\boldsymbol{\mathcal{D}}^k(\overline{\mathbf{S}}^{k,n,m,l}, \overline{\theta}^{k,n,m,l}) - \mathbf{D}(\overline{\boldsymbol{u}}^{k,n,m,l})) : \boldsymbol{\tau} = 0 && \forall\, \boldsymbol{\tau} \in \Sigma^n,\\
&\int_\Omega \partial_t \tilde{u}^{k,n,m,l} \cdot \boldsymbol{v} + \alpha \int_{\partial\Omega} \overline{\boldsymbol{u}}^{k,n,m,l} \cdot \boldsymbol{v} - \int_\Omega \overline{p}^{k,n,m,l} \operatorname{div} \boldsymbol{v} \\
&\qquad + \int_\Omega \overline{\mathbf{S}}^{k,n,m,l} : \mathbf{D}(\boldsymbol{v}) + \mathcal{B}_k(\overline{\boldsymbol{u}}^{k,n,m,l}, \overline{\boldsymbol{u}}^{k,n,m,l}, \boldsymbol{v}) = \langle \overline{\boldsymbol{f}}, \boldsymbol{v} \rangle && \forall\, \boldsymbol{v} \in V^n,\\
&- \int_\Omega q \operatorname{div} \overline{\boldsymbol{u}}^{k,n,m,l} = \frac{1}{l} \int_\Omega \nabla \overline{p}^{k,n,m,l} \cdot \nabla q && \forall\, q \in M^n,\\
&\int_\Omega \partial_t \tilde{\theta}^{k,n,m,l} \psi + \int_\Omega \kappa(\overline{\theta}^{k,n,m,l}) \nabla \overline{\theta}^{k,n,m,l} \cdot \nabla \psi + \mathcal{C}_k(\overline{\theta}^{k,n,m,l}, \overline{\boldsymbol{u}}^{k,n,m,l}, \psi) \\
&\qquad = \int_\Omega \overline{\mathbf{S}}^{k,n,m,l} : \boldsymbol{\mathcal{D}}^k(\overline{\mathbf{S}}^{k,n,m,l}, \overline{\theta}^{k,n,m,l}) \psi && \forall\, \psi \in U^m,
\end{aligned}
$$

where each equation holds everywhere in $(0,T)$. Moreover, from the discussion above we see that the following estimates hold:

$$
\begin{aligned}
&\|\overline{\boldsymbol{u}}^{k,n,m,l}\|^2_{L^\infty(0,T;L^2(\Omega))} + \tau_m \|\partial_t \tilde{\boldsymbol{u}}^{k,n,m,l}\|^2_{L^2(Q)} + \|\overline{\boldsymbol{u}}^{k,n,m,l}\|^2_{L^2(0,T;W^{1,2}(\Omega))} \\
&\quad + \|\overline{\mathbf{S}}^{k,n,m,l}\|^2_{L^2(Q)} + \|\boldsymbol{\mathcal{D}}^k(\overline{\mathbf{S}}^{k,n,m,l}, \overline{\theta}^{k,n,m,l})\|^2_{L^2(Q)} + \frac{1}{l} \|\overline{p}^{k,n,m,l}\|^2_{L^2(0,T;W^{1,2}(\Omega))} \leq c,
\end{aligned} \tag{5.20a}
$$

$$
\|\overline{\theta}^{k,n,m,l}\|^2_{L^\infty(0,T;L^2(\Omega))} + \tau_m \|\partial_t \tilde{\theta}^{k,n,m,l}\|^2_{L^2(Q)} + \|\nabla \overline{\theta}^{k,n,m,l}\|^2_{L^2(Q)} \leq c_n, \tag{5.20b}
$$

where c_n blows up as $n \to \infty$. Hence, for a (not relabelled) subsequence we have as $m \to \infty$ that

$$
\begin{aligned}
\overline{\boldsymbol{u}}^{k,n,m,l} &\overset{*}{\rightharpoonup} \boldsymbol{u}^{k,n,l} && \text{weakly* in } L^\infty(0,T;L^2(\Omega)^d),\\
\overline{\boldsymbol{u}}^{k,n,m,l} &\rightharpoonup \boldsymbol{u}^{k,n,l} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)^d),\\
\overline{\mathbf{S}}^{k,n,m,l} &\rightharpoonup \mathbf{S}^{k,n,l} && \text{weakly in } L^2(Q)^{d\times d},\\
\overline{p}^{k,n,m,l} &\rightharpoonup p^{k,n,l} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)),\\
\boldsymbol{\mathcal{D}}^k(\overline{\mathbf{S}}^{k,n,m,l}, \overline{\theta}^{k,n,m,l}) &\rightharpoonup \mathbf{D}^{k,n,l} && \text{weakly in } L^2(Q)^{d\times d},\\
\overline{\theta}^{k,n,m,l} &\overset{*}{\rightharpoonup} \theta^{k,n,l} && \text{weakly* in } L^\infty(0,T;L^2(\Omega)),\\
\overline{\theta}^{k,n,m,l} &\rightharpoonup \theta^{k,n,l} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)),
\end{aligned} \tag{5.21}
$$

In order to obtain strong convergence for the sequence of approximate temperatures, Simon's lemma will be employed. Noticing that the integrand is piecewise constant and only nonzero on $(t_j - \varepsilon, t_j]$, for $j \in \{1, \dots, T/\tau_m\}$, we have

$$\int_0^{T-\varepsilon} \|\overline{\theta}^{k,n,m,l}(s+\varepsilon,\cdot) - \overline{\theta}^{k,n,m,l}(s,\cdot)\|^2_{L^2(\Omega)} = \varepsilon \sum_{j=2}^{T/\tau_m} \|\theta_j^{k,n,m,l} - \theta_{j-1}^{k,n,m,l}\|^2_{L^2(\Omega)} \le \varepsilon c_n \xrightarrow[\varepsilon \to 0]{} 0,$$

where c_n is the constant appearing in Lemma 5.1.3. Since $\overline{\theta}^{k,n,m,l}$ is bounded uniformly both in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;W^{1,2}(\Omega))$ then Simon's lemma implies that, as $m \to \infty$,

$$\overline{\theta}^{k,n,m,l} \to \theta^{k,n,l} \qquad \text{strongly in } L^2(Q), \tag{5.22a}$$

and since $(\tilde{\theta}^{k,n,m,l} - \overline{\theta}^{k,n,m,l})$ both converge to zero in $L^2(Q)$ as $m \to \infty$ (as a consequence of (5.20)), we also obtain

$$\tilde{\theta}^{k,n,m,l} \to \theta^{k,n,l} \qquad \text{strongly in } L^2(Q). \tag{5.22b}$$

An identical argument can be applied to the sequence of approximate velocities, and therefore, as $m \to \infty$,

$$\begin{aligned}
\tilde{\boldsymbol{u}}^{k,n,m,l} &\to \boldsymbol{u}^{k,n,l} && \text{strongly in } L^2(Q)^d, \\
\overline{\boldsymbol{u}}^{k,n,m,l} &\to \boldsymbol{u}^{k,n,l} && \text{strongly in } L^2(0,T;W^{1,2}(\Omega)^d), \\
\overline{\boldsymbol{u}}^{k,n,m,l} &\to \boldsymbol{u}^{k,n,l} && \text{strongly in } L^2(0,T;L^2(\partial\Omega)^d);
\end{aligned} \tag{5.23}$$

here we used the equivalence of norms in finite-dimensional spaces, and the fact that from the estimate (5.20) we also get that $(\tilde{\boldsymbol{u}}^{k,n,m} - \overline{\boldsymbol{u}}^{k,n,m})$ converges to the zero in $L^2(Q)^d$.

The convergence properties (5.21), (5.22), and (5.23) then allow passage to the limit and one obtains that the limiting functions satisfy

$$\int_0^T \int_\Omega (\mathbf{D}^{k,n,l} - \mathbf{D}(\boldsymbol{u}^{k,n,l})) : \boldsymbol{\tau}\phi = 0 \qquad \forall\, \boldsymbol{\tau} \in \Sigma^n, \phi \in C_0^\infty(0,T),$$

$$\begin{aligned}
-\int_0^T \int_\Omega \boldsymbol{u}^{k,n,l} \cdot \boldsymbol{v}\partial_t\phi - \int_\Omega P_V^n \boldsymbol{u}_0 \cdot \boldsymbol{v}\phi(0) + \alpha \int_0^T \int_{\partial\Omega} \boldsymbol{u}^{k,n,l} \cdot \boldsymbol{v}\phi \\
+ \int_0^T \mathcal{B}_k(\boldsymbol{u}^{k,n,l}, \boldsymbol{u}^{k,n,l}, \boldsymbol{v})\phi - \int_0^T \int_\Omega p^{k,n,l} \operatorname{div} \boldsymbol{v}\phi \\
+ \int_0^T \int_\Omega \mathbf{S}^{k,n,l} : \mathbf{D}(\boldsymbol{v})\phi = \int_0^T \langle \boldsymbol{f}, \boldsymbol{v}\rangle\phi \qquad \forall\, \boldsymbol{v} \in V^n, \phi \in C_0^\infty[0,T),
\end{aligned}$$

$$-\int_0^T \int_\Omega q \operatorname{div} \boldsymbol{u}^{k,n,l}\phi = \frac{1}{l} \int_0^T \int_\Omega \nabla p^{k,n,l} \cdot \nabla q \qquad \forall\, q \in M^n, \phi \in C_0^\infty(0,T),$$

$$
-\int_0^T\int_\Omega \theta^{k,n,l}\psi\partial_t\phi - \int_\Omega \theta_0^n\psi\phi + \int_0^T \mathcal{C}_k(\theta^{k,n,l},\boldsymbol{u}^{k,n,l},\psi)\phi
$$
$$
+\int_0^T\int_\Omega \kappa(\theta^{k,n,l})\nabla\theta^{k,n,l}\cdot\nabla\psi\phi = \int_0^T\int_\Omega \mathbf{S}^{k,n,l}:\mathbf{D}^{k,n,l}\psi\phi \quad \forall\,\psi\in C^\infty(\Omega),\ \phi\in C_0^\infty[0,T).
$$

Since $n\in\mathbb{N}$ is fixed, the initial conditions can be identified in the usual way, i.e. we have that $\theta^{k,n,l}(0,\cdot)=\theta_0^n(\cdot)$ and $\boldsymbol{u}^{k,n,l}(0,\cdot)=P_V^n\boldsymbol{u}_0(\cdot)$. In addition, since we have strong convergence of the velocity gradients and the temperature in $L^2(Q)$, and thus almost everywhere convergence up to a subsequence, the constitutive relation can also be identified in a straightforward manner using Vitali's theorem (recall that $\boldsymbol{\mathcal{D}}^k$ is continuous) and so

$$
\mathbf{D}^{k,n,l}=\boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,n,l},\theta^{k,n,l}) \qquad \text{a.e. in } Q. \tag{5.24}
$$

The next step is to take the limit $n\to\infty$, and so uniform estimates in n are needed. The weak lower semicontinuity of the norms and the estimate (5.20a) and (2.9) yield

$$
\begin{aligned}
&\|\boldsymbol{u}^{k,n,l}\|_{L^\infty(0,T;L^2(\Omega))}+\|\boldsymbol{u}^{k,n,l}\|_{L^2(0,T;W^{1,2}(\Omega))}+\|\boldsymbol{u}^{k,n,l}\|_{L^{\frac{2(d+2)}{d}}(Q)}\\
&\quad+\|\mathbf{S}^{k,n,l}\|_{L^2(Q)}^2+\|\boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,n,l},\theta^{k,n,l})\|_{L^2(Q)}+\frac{1}{l}\|p^{k,n,l}\|_{L^2(0,T;W^{1,2}(\Omega))}\le c.
\end{aligned} \tag{5.25}
$$

Simon's lemma will now be applied to obtain strong convergence for the sequence of approximate velocities. To that end, observe that for any $\boldsymbol{v}\in V^n$ one has

$$
\begin{aligned}
&\int_\Omega(\boldsymbol{u}^{k,n,l}(s+\varepsilon,\cdot)-\boldsymbol{u}^{k,n,l}(s,\cdot))\cdot\boldsymbol{v}(x)\,\mathrm{d}x=\int_s^{s+\varepsilon}\int_\Omega\langle\partial_t\boldsymbol{u}^{k,n,l}(t,\cdot),\boldsymbol{v}(\cdot)\rangle\,\mathrm{d}t\\
&\le\int_s^{s+\varepsilon}\Big[\alpha\|\boldsymbol{u}^{k,n,l}(t,\cdot)\|_{L^2(\partial\Omega)}\|\boldsymbol{v}\|_{L^2(\partial\Omega)}+\|p^{k,n,l}(t,\cdot)\|_{L^2(\Omega)}\|\operatorname{div}\boldsymbol{v}\|_{L^2(\Omega)}\\
&\quad\|\mathbf{S}^{k,n,l}(t,\cdot)\|_{L^2(\Omega)}\|\mathbf{D}(\boldsymbol{v})\|_{L^2(\Omega)}+c_k\|\boldsymbol{u}^{k,n,l}(t,\cdot)\|^2_{L^{\frac{2(d+2)}{d}}(\Omega)}\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}\\
&\quad+\|\boldsymbol{f}(t,\cdot)\|_{W_{\boldsymbol{n}}^{-1,2}(\Omega)}\|\boldsymbol{v}\|_{W^{1,2}(\Omega)}\Big]\,\mathrm{d}t\le c_k(\varepsilon^{\frac{1}{2}}+\varepsilon^{\frac{2}{d+2}})\|\boldsymbol{v}\|_{W^{1,2}(\Omega)}.
\end{aligned}
$$

Letting $\boldsymbol{v}=\boldsymbol{u}^{k,n,l}(s+\varepsilon,\cdot)-\boldsymbol{u}^{k,n,l}(s,\cdot)$ and integrating with respect to s we then see that

$$
\int_0^{T-\varepsilon}\|\boldsymbol{u}^{k,n,l}(s+\varepsilon,\cdot)-\boldsymbol{u}^{k,n,l}(s,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}s\overset{(5.25)}{\le}c_k(\varepsilon^{\frac{1}{2}}+\varepsilon^{\frac{2}{d+2}})\xrightarrow[\varepsilon\to0]{}0, \tag{5.26}
$$

uniformly in $n\in\mathbb{N}$. Hence, up to subsequences, the following convergences hold:

$$
\begin{aligned}
\boldsymbol{u}^{k,n,l}&\overset{*}{\rightharpoonup}\boldsymbol{u}^{k,l} &&\text{weakly* in } L^\infty(0,T;L^2(\Omega)^d),\\
\boldsymbol{u}^{k,n,l}&\rightharpoonup\boldsymbol{u}^{k,l} &&\text{weakly in } L^2(0,T;W^{1,2}(\Omega)^d),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{u}^{k,n,l} &\rightharpoonup \boldsymbol{u}^{k,l} && \text{weakly in } L^2(0,T;L^2(\partial\Omega)^d),\\
\boldsymbol{u}^{k,n,l} &\to \boldsymbol{u}^{k,l} && \text{strongly in } L^q(Q)^d, \text{ for } q\in[1,\tfrac{2(d+2)}{d}),\\
\mathbf{S}^{k,n,l} &\rightharpoonup \mathbf{S}^{k,l} && \text{weakly in } L^2(Q)^{d\times d},\\
p^{k,n,l} &\rightharpoonup p^{k,l} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)),\\
\boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,n,l},\theta^{k,n,l}) &\rightharpoonup \mathbf{D}^{k,l} && \text{weakly in } L^2(Q)^{d\times d}.
\end{aligned}
\tag{5.27}
$$

The convergence of the velocity traces can actually be improved to strong convergence. To see this, observe that by function space interpolation with respect to the degree of smoothness [MNRR96, Lemma 2.18], we have that

$$
\|\boldsymbol{v}\|_{W^{1-\varepsilon,2}(\Omega)} \le c\|\boldsymbol{v}\|_{W^{1,2}(\Omega)}^{1-\varepsilon}\|\boldsymbol{u}\|_{L^2(\Omega)}^{\varepsilon} \quad \forall\, \boldsymbol{v}\in W^{1,2}(\Omega)^d \cap L^2(\Omega)^d, \varepsilon\in[0,1].
$$

Integrating in time and using Hölder inequality then gives

$$
\int_0^T \|\boldsymbol{v}\|_{W^{1-\varepsilon,2}(\Omega)}^2 \,\mathrm{d}t \le c\left(\int_0^T \|\boldsymbol{v}\|_{W^{1,2}(\Omega)}^2\right)^{1-\varepsilon}\left(\int_0^T \|\boldsymbol{u}\|_{L^2(\Omega)}^2\,\mathrm{d}t\right)^{\varepsilon},
$$

for any $\boldsymbol{v}\in L^2(0,T;W^{1,2}(\Omega)^d)\cap L^2(Q)^d$, which implies that $\boldsymbol{u}^{k,n,l}\to\boldsymbol{u}^{k,l}$ strongly in $L^2(0,T;W^{1-\varepsilon,2}(\Omega)^d)$, for any $\varepsilon\in(0,1]$. Choosing ε small enough so that $2(1-\varepsilon)>1$, which ensures that the trace operator is well defined and bounded, we see that $W^{1-\varepsilon,2}(\Omega)^d \hookrightarrow W^{1-\varepsilon-\frac{1}{2},2}(\partial\Omega)^d \hookrightarrow L^2(\partial\Omega)^d$, which yields

$$
\boldsymbol{u}^{k,n,l}\to\boldsymbol{u}^{k,l} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)^d). \tag{5.28}
$$

Now, testing the energy equation with the function

$$
\psi^{k,n,l}(t,x) = \mathbb{1}_{(0,\tau)}(t)\min\{0,\theta^{k,n,l}(t,x)-\overline{c}\}\le 0,
$$

where $\tau\in(0,T)$ and recalling that $\theta_0^n\ge\overline{c}$ we conclude that $\theta^{k,n,l}\ge\overline{c}>0$ a.e. in Q. In addition, testing with $\psi\equiv 1$ we obtain the uniform estimate

$$
\|\theta^{k,n,l}\|_{L^\infty(0,T;L^1(\Omega))}\le c. \tag{5.29}
$$

On the other hand, setting $\psi=(\theta^{k,n,l})^\lambda$, for $-1<\lambda<0$, we get

$$
\begin{aligned}
&\int_Q \mathbf{S}^{k,n,l}:\mathbf{D}^{k,n,l}(\theta^{k,n,l})^\lambda - \int_Q \kappa(\theta^{k,n,l})\nabla\theta^{k,n,l}\cdot\nabla(\theta^{k,n,l})^\lambda\\
&= \int_0^T \mathcal{C}_k(\theta^{k,n,l},\boldsymbol{u}^{k,n,l},(\theta^{k,n,l})^\lambda) + \langle\partial_t\theta^{k,n,l},(\theta^{k,n,l})^\lambda\rangle\\
&= \frac{\|(\theta^{k,n,l}(T,\cdot))^{\lambda+1}\|_{L^1(\Omega)} - \|(\theta_0^n)^{\lambda+1}\|_{L^1(\Omega)}}{\lambda+1}\le c,
\end{aligned}
$$

where we used (5.29); this implies that

$$\int_Q |\nabla(\theta^{k,n,l})^{\frac{\lambda+1}{2}}|^2 \leq c, \tag{5.30}$$

which means that $(\theta^{k,n,l})^{\frac{\lambda+1}{2}} \in L^2(0,T;W^{1,2}(\Omega)) \hookrightarrow L^2(0,T;L^6(\Omega))$. By function space interpolation this yields that

$$\|\theta^{k,n,l}\|_{L^s(Q)} \leq c \quad \text{for every } s \in \left[1, \tfrac{5}{3}\right), \tag{5.31a}$$

which in turn implies that

$$\|\theta^{k,n,l}\|_{L^s(0,T;W^{1,s}(\Omega))} \leq c \quad \text{for every } s \in \left[1, \tfrac{5}{4}\right). \tag{5.31b}$$

From (5.31) and the equation satisfied by $\theta^{k,n,l}$ one readily sees that

$$\begin{aligned} \|\partial_t\theta^{k,n,l}\|_{L^1(0,T;(W^{1,q}(\Omega))^*)} &\leq c_k \qquad \text{for } q \in (5,\infty], \\ \|\partial_t\theta^{k,n,l}\|_{L^1(0,T;(W^{1,q}(\Omega))^*)} &\leq c \qquad \text{for } q \in (10,\infty], \end{aligned} \tag{5.32}$$

where c_k blows up as $k \to \infty$. Thus, using the Aubin–Lions lemma and the above estimates we obtain the convergences

$$\begin{aligned} \theta^{k,n,l} &\rightharpoonup \theta^{k,l} \quad \text{weakly in } L^s(0,T;W^{1,s}(\Omega)), \text{ for } s \in [1, \tfrac{5}{4}), \\ \theta^{k,n,l} &\to \theta^{k,l} \qquad\qquad \text{strongly in } L^s(Q), \text{ for } s \in [1, \tfrac{5}{3}). \end{aligned} \tag{5.33}$$

The above suffices to pass to the limit and obtain

$$\begin{aligned} &\int_\Omega (\mathbf{D}^{k,l} - \mathbf{D}(\boldsymbol{u}^{k,l})) : \boldsymbol{\tau} = 0 && \forall\, \boldsymbol{\tau} \in L^2_{\mathrm{sym}}(\Omega)^{d\times d}, \\ &\int_\Omega \partial_t \boldsymbol{u}^{k,l}\cdot\boldsymbol{v} + \alpha\int_{\partial\Omega}\boldsymbol{u}^{k,l}\cdot\boldsymbol{v} - \int_\Omega p^{k,l}\operatorname{div}\boldsymbol{v} \\ &\qquad + \int_\Omega \mathbf{S}^{k,l} : \mathbf{D}(\boldsymbol{v}) + \mathcal{B}_k(\boldsymbol{u}^{k,l},\boldsymbol{u}^{k,l},\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle && \forall\, \boldsymbol{v} \in W^{1,2}_{\boldsymbol{n}}(\Omega)^d, \\ &\int_\Omega q \operatorname{div}\boldsymbol{u}^{k,l} = \frac{1}{l}\int_\Omega \nabla p^{k,l}\cdot\nabla q && \forall\, q \in W^{1,2}(\Omega). \end{aligned}$$

The initial condition for the velocity can be identified in the standard way. We now claim that the nonlinear limit in the constitutive relation can be identified as

$$\mathbf{D}^{k,l} = \boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,l},\theta^{k,l}) \qquad \text{a.e. in } Q. \tag{5.34}$$

Indeed, since k is fixed the velocity $\boldsymbol{u}^{k,l}$ is an admissible test function in the weak formulation and therefore we have an energy identity available; this makes it straightforward to prove that

$$\limsup_{n\to\infty} \int_Q \mathbf{S}^{k,n,l} : \mathbf{D}^{k,n,l} \leq \int_Q \mathbf{S}^{k,l} : \mathbf{D}^{k,l}. \tag{5.35}$$

Moreover, from the definition of $\boldsymbol{\mathcal{D}}^k$ we see that, for any $\boldsymbol{\sigma} \in L^2(\Omega)^{d\times d}$ and $t \in \mathbb{R}$,

$$|\boldsymbol{\mathcal{D}}^k(\boldsymbol{\sigma}, t)| \overset{(5.10)}{\leq} \left(k + \frac{1}{2\mu(t)}\right) |\boldsymbol{\sigma}| \leq c_k |\boldsymbol{\sigma}|,$$

and so by the dominated convergence theorem we infer that

$$\boldsymbol{\mathcal{D}}^k(\boldsymbol{\sigma}, \theta^{k,n,l}) \to \boldsymbol{\mathcal{D}}^k(\boldsymbol{\sigma}, \theta^{k,l}) \quad \text{strongly in } L^2(\Omega)^{d\times d}, \tag{5.36}$$

as $n \to \infty$. Combining the monotonicity of $\boldsymbol{\mathcal{D}}^k$ with (5.35) and (5.36) yields, for an arbitrary $\boldsymbol{\sigma} \in L^2(\Omega)^{d\times d}$:

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \int_Q (\boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,n,l}, \theta^{k,n,l} - \boldsymbol{\mathcal{D}}^k(\boldsymbol{\sigma}, \theta^{k,n,l}))) : (\mathbf{S}^{k,n,l} - \boldsymbol{\sigma}) \\ &\leq \int_Q (\mathbf{D}^{k,l} - \boldsymbol{\mathcal{D}}^k(\boldsymbol{\sigma}, \theta^{k,l})) : (\mathbf{S}^{k,l} - \boldsymbol{\sigma}). \end{aligned}$$

Choosing $\boldsymbol{\sigma} = \mathbf{S}^{k,l} \pm \varepsilon \boldsymbol{\tau}$, with an arbitrary $\boldsymbol{\tau} \in C_0^\infty(Q)^{d\times d}$, and letting $\varepsilon \to 0$ concludes the proof of the claim (5.34).

In order to pass to the limit in the energy equation it is necessary to investigate the convergence properties of $\mathbf{S}^{k,n,l} : \mathbf{D}^{k,n,l}$ in $L^1(\Omega)$. First, from the monotonicity of $\boldsymbol{\mathcal{D}}^k$ we can get

$$\int_Q \mathbf{S}^{k,l} : \mathbf{D}^{k,l} \leq \liminf_{n\to\infty} \int_Q \boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,n,l}, \theta^{k,n,l}) : \mathbf{S}^{k,n,l},$$

and so by (5.35) the equality actually holds:

$$\int_Q \mathbf{D}^{k,l} : \mathbf{S}^{k,l} = \lim_{n\to\infty} \int_Q \boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,n,l}, \theta^{k,n,l}) : \mathbf{S}^{k,n,l},$$

which in turn implies that (note that this function is non-negative)

$$(\boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,n,l}, \theta^{k,n,l}) - \boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,l}, \theta^{k,l})) : (\mathbf{S}^{k,n,l} - \mathbf{S}^{k,l}) \to 0 \quad \text{strongly in } L^1(\Omega). \tag{5.37}$$

Writing the product as

$$\begin{aligned} \boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,n,l}, \theta^{k,n,l}) : \mathbf{S}^{k,n,l} &= \boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,n,l}, \theta^{k,n,l}) : \mathbf{S}^{k,l} + \boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,l}, \theta^{k,l}) : (\mathbf{S}^{k,n,l} - \mathbf{S}^{k,l}) \\ &\quad + (\boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,n,l}, \theta^{k,n,l}) - \boldsymbol{\mathcal{D}}^k(\mathbf{S}^{k,l}, \theta^{k,l})) : (\mathbf{S}^{k,n,l} - \mathbf{S}^{k,l}), \end{aligned}$$

and using (5.27) immediately yields that $\mathbf{S}^{k,n,l} : \mathbf{D}^{k,n,l} \rightharpoonup \mathbf{S}^{k,l} : \mathbf{D}^{k,l}$ weakly in $L^1(Q)$ as $n \to \infty$, which allows one to pass to the limit in the energy equation:

$$\int_\Omega \partial_t \theta^{k,l} \psi + \int_\Omega \kappa(\theta^{k,l}) \nabla \theta^{k,l} \cdot \nabla \psi + \mathcal{C}_k(\theta^{k,l}, \boldsymbol{u}^{k,l}, \psi) = \int_\Omega \mathbf{S}^{k,l} : \mathbf{D}^{k,l} \psi \quad \forall\, \psi \in W^{1,\infty}(\Omega).$$

Let us now turn to the quasi-compressibility limit $l \to \infty$. First, notice that Fatou's lemma and the weak lower-semicontinuity of the norms result in the a priori estimates

$$
\begin{aligned}
&\|\boldsymbol{u}^{k,l}\|_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u}^{k,l}\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\boldsymbol{u}^{k,l}\|_{L^{\frac{2(d+2)}{d}}(Q)} + \|\mathbf{S}^{k,l}\|_{L^2(Q)} + \|\mathbf{D}^{k,l}\|_{L^2(Q)} \\
&\quad + \frac{1}{l}\|p^{k,l}\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\theta^{k,l}\|_{L^s(Q)} + \|\theta^{k,l}\|_{L^q(0,T;W^{1,q}(\Omega))} + \|\theta^{k,l}\|_{L^\infty(0,T;L^1(\Omega))} \le c,
\end{aligned} \tag{5.38}
$$

for $s \in [1, \frac{5}{3})$, $q \in [1, \frac{5}{4})$. Before we can proceed further, uniform estimates for the pressure are also required. Consider the following auxiliary problem defined for almost every $t \in (0,T)$ and some $\beta \in (1,2]$:

$$
\begin{aligned}
\Delta h^{k,l} &= |p^{k,l}|^{\beta-2}p^{k,l} - \frac{1}{|\Omega|}\int_\Omega |p^{k,l}|^{\beta-2}p^{k,l} \quad \text{ in } \Omega, \\
\nabla h^{k,l}\cdot \boldsymbol{n} &= 0 \quad \text{ on } \partial\Omega, \quad \int_\Omega h^{k,l} = 0.
\end{aligned} \tag{5.39}
$$

Suppose for now that the Neumann problem (5.39) is $W^{2,\beta'}$-regular for any $\beta \in (1,2]$ (in the end we will just pick two particular values for β); this means that we have the estimate

$$
\|h^{k,l}\|^{\beta'}_{W^{2,\beta'}(\Omega)} \le c\|p^{k,n}\|^{\beta}_{L^\beta(\Omega)}, \tag{5.40}
$$

where c is independent of k, l, and so testing the momentum equation with $\nabla h^{k,l}$ yields

$$
\begin{aligned}
\int_0^T \|p^{k,l}(t,\cdot)\|^\beta_{L^\beta(\Omega)}\,\mathrm{d}t &= \int_Q \mathbf{S}^{k,l} : \mathbf{D}(\nabla h^{k,l}) - \int_0^T \langle \boldsymbol{f}, \nabla h^{k,l}\rangle + \alpha\int_0^T\int_{\partial\Omega} \boldsymbol{u}^{k,l}\cdot\nabla h^{k,l} \\
&\quad + \int_0^T \mathcal{B}_k(\boldsymbol{u}^{k,l}, \boldsymbol{u}^{k,l}, \nabla h^{k,l}) + \int_Q \partial_t \boldsymbol{u}^{k,l}\cdot\nabla h^{k,l} \\
&=: I_1 + \cdots + I_5.
\end{aligned}
$$

Now let $\eta > 0$ be arbitrary. The first three terms can be dealt with easily using the estimate (5.38) and Young's inequality:

$$
\begin{aligned}
I_1 + I_2 + I_3 &\le c(\eta)(\|\mathbf{S}^{k,l}\|^2_{L^2(Q)} + \|\boldsymbol{f}\|^2_{L^2(0,T;W^{-1,2}_{\boldsymbol{n}}(\Omega))} + \|\boldsymbol{u}^{k,l}\|^2_{L^2(0,T;L^2(\partial\Omega))}) + \eta\int_0^T \|p^{k,l}\|^\beta_{L^\beta(\Omega)} \\
&\le c(\eta) + \eta\int_0^T \|p^{k,l}\|^\beta_{L^\beta(\Omega)} \qquad \text{for any } \beta \in (1,2].
\end{aligned}
$$

In order to estimate I_4 we have two options:

$$
I_4 \le c(\eta)\|((\phi_k \boldsymbol{u}^{k,l}) * \omega_{1/k})_{\mathrm{div}}\|^2_{L^{d+2}(Q)}\|\boldsymbol{u}^{k,l}\|^2_{L^{\frac{2(d+2)}{d}}(Q)} + \eta\int_0^T \|p^{k,l}\|^\beta_{L^\beta(\Omega)}
$$

$$
\begin{aligned}
&\leq c_k(\eta) + \eta \int_0^T \|p^{k,l}\|^{\beta}_{L^{\beta}(\Omega)} \quad \text{for any } \beta \in (1,2], \\
I_4 &\leq c(\eta) \|\boldsymbol{u}^{k,l}\|^{\frac{2(d+2)}{d}}_{L^{\frac{2(d+2)}{d}}(Q)} + \eta \int_0^T \|p^{k,l}\|^{\beta}_{L^{\beta}(\Omega)} \\
&\leq c(\eta) + \eta \int_0^T \|p^{k,l}\|^{\beta}_{L^{\beta}(\Omega)} \quad \text{for any } \beta \in (1, \check{2}].
\end{aligned}
$$

We claim now that $I_5 \leq 0$. To see this, let $\{\boldsymbol{u}_i^{k,l}\}_{i\in\mathbb{N}}$ be a sequence of smooth functions such that $\boldsymbol{u}_i^{k,l} \to \boldsymbol{u}^{k,l}$ strongly in $L^2(0,T;W^{1,2}(\Omega))$ and $\partial_t \boldsymbol{u}_i^{k,l} \to \partial_t \boldsymbol{u}_i^{k,l}$ strongly in $L^2(0,T;W^{-1,2}_{\boldsymbol{n}}(\Omega))$. Define $p_i^{k,l}$ and $h_i^{k,l}$ as the solutions of the problems

$$
\begin{cases}
\frac{1}{l}\Delta p_i^{k,l} = \operatorname{div} \boldsymbol{u}_i^{k,l} & \text{in } \Omega, \\
\nabla p_i^{k,l} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \\
\int_\Omega p_i^{k,l} = 0.
\end{cases}
\qquad
\begin{cases}
\Delta h_i^{k,l} = |p_i^{k,l}|^{\beta-2} p_i^{k,l} - \frac{1}{|\Omega|} \int_\Omega |p_i^{k,l}|^{\beta-2} p_i^{k,l} & \text{in } \Omega, \\
\nabla h_i^{k,l} = 0 & \text{on } \partial\Omega, \\
\int_\Omega h_i^{k,l} = 0,
\end{cases}
$$

where $\beta \in (1,2]$ is arbitrary. Hence, we can write

$$
I_5 = \lim_{i\to\infty} I_5^i := \lim_{i\to\infty} \int_0^T \langle \partial_t \boldsymbol{u}_i^{k,l}, \nabla h_i^{k,l} \rangle.
$$

Decomposing into $\boldsymbol{u}_i^{k,l} = \boldsymbol{w}_{i,\text{div}}^{k,l} + \nabla \phi_i^{k,l}$, where $\boldsymbol{w}_{i,\text{div}}^{k,l}$ is divergence-free, and noting that the uniqueness of the Helmholtz decomposition implies that $\frac{1}{l}\phi_i^{k,l} = p_i^{k,l}$, we obtain

$$
\begin{aligned}
I_5^i &= \int_0^T \langle \partial_t \nabla \phi_i^{k,l}, \nabla h_i^{k,l} \rangle = -\int_Q \partial_t \phi_i^{k,l} \Delta h_i^{k,l} = -\frac{1}{l} \int_Q |p_i^{k,l}|^{\beta-2} p_i^{k,l} \partial_t p_i^{k,l} \\
&= -\frac{1}{l\beta} \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \|p_i^{k,l}(t,\cdot)\|^{\beta}_{L^{\beta}(\Omega)} = -\frac{1}{l\beta} \|p_i^{k,l}(T,\cdot)\|^{\beta}_{L^{\beta}(\Omega)} + \frac{1}{l\beta} \|p_i^{k,l}(0,\cdot)\|^{\beta}_{L^{\beta}(\Omega)},
\end{aligned}
$$

and recalling that $\boldsymbol{u}^{k,l}(0,\cdot) = \boldsymbol{u}_0 \in L^2_{\text{div}}(\Omega)^d$, we conclude that $I_5 \leq 0$. Consequently, using $\beta = 2$ and $\beta = \check{2}$ we obtain the following estimates for the pressure

$$
\begin{aligned}
\|p^{k,l}\|_{L^2(Q)} &\leq c_k, \\
\|p^{k,l}\|_{L^{\check{2}}(Q)} &\leq c.
\end{aligned}
\tag{5.41}
$$

Observing that $2, \check{2}' \in [2,3]$, then the required regularity estimate to obtain (5.41) is guaranteed for $d = 3$ by Lemma 2.1.1 if Ω is taken as a convex polyhedron such that the angles at the edges $\{\varpi_i\}_{i=1}^{M_e}$ satisfy

$$
\frac{\pi}{\varpi_i} > 2 - \frac{2}{\check{2}'} \quad \text{for all } i \in \{1, \ldots, M_e\}. \tag{5.42}
$$

The estimates (5.41) and the momentum equation then imply that

$$\|\partial_t \boldsymbol{u}^{k,l}\|_{L^2(0,T;W_{\boldsymbol{n}}^{-1,2}(\Omega)^d)} \le c_k. \tag{5.43}$$

Hence, by the Aubin–Lions lemma we have (up to subsequences), as $l \to \infty$:

$$\begin{aligned}
\boldsymbol{u}^{k,l} &\overset{*}{\rightharpoonup} \boldsymbol{u}^k && \text{weakly* in } L^\infty(0,T;L^2(\Omega)^d),\\
\boldsymbol{u}^{k,l} &\rightharpoonup \boldsymbol{u}^k && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)^d),\\
\boldsymbol{u}^{k,l} &\to \boldsymbol{u}^k && \text{strongly in } L^2(0,T;L^2(\partial\Omega)^d),\\
\boldsymbol{u}^{k,l} &\to \boldsymbol{u}^k && \text{strongly in } L^q(Q)^d, \text{ for } q \in [1, \tfrac{2(d+2)}{d}),\\
p^{k,l} &\rightharpoonup p^k && \text{weakly in } L^2(Q),\\
\mathbf{S}^{k,l} &\rightharpoonup \mathbf{S}^k && \text{weakly in } L^2(Q)^{d\times d},\\
\mathbf{D}^{k,l} &\rightharpoonup \mathbf{D}^k && \text{weakly in } L^2(Q)^{d\times d},\\
\theta^{k,l} &\rightharpoonup \theta^k && \text{weakly in } L^q(0,T;W^{1,q}(\Omega)), \text{ for } q \in [1, \tfrac{5}{4}),\\
\theta^{k,l} &\to \theta^k && \text{strongly in } L^s(Q), \text{ for } s \in [1, \tfrac{5}{3}).
\end{aligned} \tag{5.44}$$

Using the estimate (5.38) we can prove that the limiting function $\boldsymbol{u}^k$ is actually divergence-free. To do so, let $q \in L^2(0,T;W^{1,2}(\Omega))$ be arbitrary, then

$$\begin{aligned}
\left|\int_Q q \operatorname{div} \boldsymbol{u}^k\right| &= \lim_{l\to\infty}\left|\int_Q q \operatorname{div} \boldsymbol{u}^{k,l}\right| = \lim_{l\to\infty}\left|\frac{1}{l}\int_Q \nabla q \cdot \nabla p^{k,l}\right|\\
&\le c \lim_{l\to\infty} \frac{1}{l^{1/2}} \|q\|_{L^2(0,T;W^{1,2}(\Omega))} = 0.
\end{aligned}$$

Thus, the limiting functions satisfy the system

$$\begin{aligned}
\int_\Omega (\mathbf{D}^k - \mathbf{D}(\boldsymbol{u}^k)) : \boldsymbol{\tau} = 0 \qquad & \forall\, \boldsymbol{\tau} \in L^2_{\text{sym}}(\Omega)^{d\times d},\\
\int_\Omega \partial_t \boldsymbol{u}^k \cdot \boldsymbol{v} + \alpha \int_{\partial\Omega} \boldsymbol{u}^k \cdot \boldsymbol{v} - \int_\Omega p^k \operatorname{div} \boldsymbol{v} + \int_\Omega \mathbf{S}^k : \mathbf{D}(\boldsymbol{v}) & \\
+\mathcal{B}_k(\boldsymbol{u}^k, \boldsymbol{u}^k, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \qquad & \forall\, \boldsymbol{v} \in W^{1,2}_{\boldsymbol{n}}(\Omega)^d,\\
\int_\Omega q \operatorname{div} \boldsymbol{u}^k = 0 \qquad & \forall\, q \in L^2(\Omega).
\end{aligned}$$

At this level the velocity $\boldsymbol{u}^k$ is still an admissible test function in the momentum equation, and so there is still an energy identity available. This means that the same argument used to obtain (5.34) is applicable here and hence

$$\mathbf{D}^k = \boldsymbol{\mathcal{D}}^k(\mathbf{S}^k, \theta^k) \qquad \text{a.e. in } Q, \tag{5.45}$$

and

$$\mathbf{S}^{k,l} : \mathbf{D}^{k,l} \rightharpoonup \mathbf{S}^k : \mathbf{D}^k \qquad \text{weakly in } L^1(Q). \tag{5.46}$$

This allows passage to the limit in the energy equation:

$$\int_\Omega \partial_t \theta^k \psi + \int_\Omega \kappa(\theta^k)\nabla\theta^k \cdot \nabla\psi - \int_\Omega \theta^k \boldsymbol{u}^k \cdot \nabla\psi = \int_\Omega \mathbf{S}^{k,l} : \mathbf{D}^{k,l}\psi \quad \forall\, \psi \in W^{1,\infty}(\Omega).$$

Now, the weak lower semicontinuity of norms, Fatou's lemma and function space interpolation result in the following uniform estimate

$$\begin{aligned} &\|\boldsymbol{u}^k\|_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u}^k\|_{L^2(0,T;W^{1,2}(\Omega))} + \|\boldsymbol{u}^k\|_{L^{\frac{2(d+2)}{d}}(Q)} + \|\mathbf{S}^k\|^2_{L^2(Q)} + \|\mathbf{D}^k\|_{L^2(Q)} \\ &\quad + \|p^k\|_{L^{\check{2}}(Q)} + \|\theta^k\|_{L^s(Q)} + \|\theta^k\|_{L^q(0,T;W^{1,q}(\Omega))} + \|\theta^k\|_{L^\infty(0,T;L^1(\Omega))} \le c, \end{aligned} \tag{5.47}$$

for any $s \in [1, \frac{5}{3})$ and $q \in [1, \frac{5}{4})$. In addition, using the weak formulation we can also estimate the time derivatives

$$\|\partial_t \theta^k\|_{L^1(0,T;(W^{1,m}(\Omega))^*)} + \|\partial_t \boldsymbol{u}^k\|_{L^{\check{2}}(0,T;W^{-1,\check{2}}_{\boldsymbol{n}}(\Omega)^d)} \le c, \tag{5.48}$$

for $m \in (10, \infty)$. Hence, up to subsequences, the following convergences hold as $k \to \infty$:

$$\begin{aligned} \boldsymbol{u}^k &\overset{*}{\rightharpoonup} \boldsymbol{u} && \text{weakly* in } L^\infty(0,T;L^2(\Omega)^d), \\ \boldsymbol{u}^k &\rightharpoonup \boldsymbol{u} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)^d), \\ \boldsymbol{u}^k &\to \boldsymbol{u} && \text{strongly in } L^2(0,T;L^2(\partial\Omega)^d), \\ \partial_t\boldsymbol{u}^k &\rightharpoonup \partial_t\boldsymbol{u} && \text{weakly in } L^{\check{2}}(0,T;W^{-1,\check{2}}(\Omega)^d), \\ \boldsymbol{u}^k &\to \boldsymbol{u} && \text{strongly in } L^q(Q)^d, \text{ for } q \in [1, \tfrac{2(d+2)}{d}), \\ p^k &\rightharpoonup p && \text{weakly in } L^{\check{2}}(Q), \\ \mathbf{S}^k &\rightharpoonup \mathbf{S} && \text{weakly in } L^2(Q)^{d\times d}, \\ \mathbf{D}^k &\rightharpoonup \mathbf{D} && \text{weakly in } L^2(Q)^{d\times d}, \\ \theta^k &\rightharpoonup \theta && \text{weakly in } L^q(0,T;W^{1,q}(\Omega)), \text{ for } q \in [1, \tfrac{5}{4}), \\ \theta^k &\to \theta && \text{strongly in } L^s(Q), \text{ for } s \in [1, \tfrac{5}{3}), \\ \partial_t\theta^k &\overset{*}{\rightharpoonup} \partial_t\theta && \text{weakly* in } \mathcal{M}(0,T;(W^{1,m}(\Omega))^*), \text{ for } q \in (10, \infty). \end{aligned} \tag{5.49}$$

Consequently, we obtain that the stress $\mathbf{S}$, velocity $\boldsymbol{u}$ and pressure p satisfy the following system almost everywhere in $(0,T)$:

$$\begin{aligned} \int_\Omega (\mathbf{D} - \mathbf{D}(\boldsymbol{u})) : \boldsymbol{\tau} &= 0 && \forall\, \boldsymbol{\tau} \in L^2_{\mathrm{sym}}(\Omega)^{d\times d}, \\ \int_\Omega \partial_t \boldsymbol{u} \cdot \boldsymbol{v} + \alpha \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{v} - \int_\Omega p \operatorname{div} \boldsymbol{v} + \int_\Omega \mathbf{S} : \mathbf{D}(\boldsymbol{v}) & \\ - \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v}) &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle && \forall\, \boldsymbol{v} \in W^{1,\check{2}'}_{\boldsymbol{n}}(\Omega)^d, \\ \int_\Omega q \operatorname{div} \boldsymbol{u} &= 0 && \forall\, q \in L^2(\Omega). \end{aligned}$$

If $d=2$ then $\check{2}=2$, which means that an energy identity is still available for $\boldsymbol{u}$ and thus the same argument used to obtain (5.34) can be applied here to identify the constitutive relation and pass to the limit in the temperature equation. On the other hand, in the three-dimensional case $\check{2}<2$ and so admissibility of the velocity in the weak formulation is lost and it is not possible to pass to the limit in the temperature equation in its current form; an equation for the total energy must be used instead. Define the sequence of functions

$$E^k := \frac{|\boldsymbol{u}^k|^2}{2} + \theta^k.$$

Testing the equation for $\boldsymbol{u}^k$ with $v = \boldsymbol{u}^k\psi$, where $\psi \in C_0^\infty([0,T); W^{1,\infty}(\Omega))$ is arbitrary, adding the result to the equation for the temperature θ^k and using integration by parts yields:

$$-\int_Q E^k\partial_t\psi - \int_\Omega \left(\frac{|\boldsymbol{u}_0|^2}{2} + \theta_0\right)\psi + \int_Q \left(\mathbf{S}^k\boldsymbol{u}^k - p^k\boldsymbol{u}^k - E^k((\phi_k\boldsymbol{u}^k) * \omega_{1/k})\right)\cdot\nabla\psi$$
$$+\int_Q \kappa(\theta^k)\nabla\theta^k\cdot\nabla\psi + \alpha\int_0^T\int_{\partial\Omega}|\boldsymbol{u}^k|^2\psi = \int_0^T\langle \boldsymbol{f}, \boldsymbol{u}^k\psi\rangle \quad \forall\,\psi\in C_0^\infty([0,T);W^{1,\infty}(\Omega)).$$

The advantage of using this formulation is that now all the terms allow passage to the limit, and we find therefore that the total energy satisfies

$$\langle\partial_t E,\psi\rangle + \int_\Omega(\mathbf{S}\boldsymbol{u} - (E+p)\boldsymbol{u} + \kappa(\theta)\nabla\theta)\cdot\nabla\psi + \alpha\int_{\partial\Omega}|\boldsymbol{u}|^2\psi = \int_0^T\langle\boldsymbol{f},\boldsymbol{u}\psi\rangle,$$

almost everywhere in $(0,T)$ and for every $\psi\in W^{1,\infty}(\Omega)$. From the equation then one reads that

$$\|\partial_t E\|_{L^1(0,T;(W^{1,q'}(\Omega))^*)} \leq c \quad \text{for } q\in[1,\tfrac{10}{9}). \tag{5.50}$$

The next step is to identify the nonlinear limit in the constitutive relation, i.e. prove that

$$(\mathbf{D},\mathbf{S},\theta)\in\mathcal{A} \quad \text{a.e. in } Q. \tag{5.51}$$

To do so we will employ Lemma 5.1.2, which requires obtaining an inequality like (5.35). Since we do not have an energy identity at our disposal, we will show instead that

$$\limsup_{k\to\infty}\int_{E_j}\mathbf{S}^k:\mathbf{D}^k \leq \int_{E_j}\mathbf{S}:\mathbf{D}, \tag{5.52}$$

where $E_j\subset Q$ is a sequence of measurable sets such that $|Q\backslash E_j|\to 0$ as $j\to\infty$. In the proof of Theorem 3.1.5 this problem was tackled using a Lipschitz truncation of the velocity error $\boldsymbol{u}^k-\boldsymbol{u}$; such a truncation was necessary because of the potentially very low integrability of the term $\boldsymbol{u} \otimes \boldsymbol{u}$. In contrast, for the problem under consideration in this chapter we actually have that the convective term $\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) = \boldsymbol{u} \cdot \nabla \boldsymbol{u}$ is integrable (since $2 > 2\frac{d+1}{d+2}$) and so an L^∞-truncation will suffice in this case.

At the moment we only know that the pressure belongs to $L^{\check{2}}(\Omega)$ which means that we cannot yet test with an L^∞-truncation of the velocity error. As a preliminary step we need to perform a decomposition of the pressure into an L^2-integrable component and a component in a Bochner–Sobolev space. First define the function p_1^k as the solution of the problem

$$\int_\Omega p_1^k(t,\cdot)\Delta\phi = \int_\Omega \mathbf{S}^k(t,\cdot) \cdot \nabla^2\phi + \alpha \int_{\partial\Omega} \boldsymbol{u}^k(t,\cdot) \cdot \nabla\phi \;\forall\, \phi \in W^{2,2}(\Omega), \nabla\phi \in W^{1,2}_{\boldsymbol{n}}(\Omega)^d. \tag{5.53}$$

The problem above is well defined due to the $W^{2,2}$-regularity of the Neumann problem for the Laplacian. Defining now $p_2^k := p^k - p_1^k$, we see from the momentum equation that p_2^k satisfies

$$-\int_\Omega \nabla p_2^k(t,\cdot)\cdot\nabla\phi = \int_\Omega \operatorname{div}(((\phi_k \boldsymbol{u}^k)*\omega_{1/k})\otimes\boldsymbol{u}^k)\cdot\nabla\phi \quad \forall\, \phi \in W^{2,2}(\Omega), \nabla\phi \in W^{1,2}_{\boldsymbol{n}}(\Omega)^d. \tag{5.54}$$

From elliptic regularity estimates (c.f. [Maz09, Dau92], and see e.g. the derivation of estimates (5.41)) we obtain that

$$\|p_1^k\|_{L^2(Q)} + \|p_2^k\|_{L^{\frac{2(d+2)}{2d+2}}(0,T;W^{1,\frac{2(d+2)}{2d+2}}(\Omega))} \leq c. \tag{5.55}$$

Let us now set up the L^∞-truncation; we will follow the approach presented in [MZ18]. First, define the sequence of functions

$$I^k := |p_1^k|^2 + |\nabla\boldsymbol{u}^k|^2 + |\nabla\boldsymbol{u}|^2 + |\mathbf{S}|^2 + |\mathbf{S}^k|^2,$$

and for an arbitrary $N \in \mathbb{N}$ define the sets

$$Q_i^k := \{N^i < |\boldsymbol{u}^k - \boldsymbol{u}| < N^{i+1}\} \quad \text{for } i \in \{1,\dots,N\}.$$

Note that $Q_i^k \cap Q_j^k = \emptyset$ and so

$$\sum_{i=1}^N \int_{Q_i^k} I^k = \int_Q I^k \leq c,$$

which means that for every $k \in \mathbb{N}$ there is an $i_k \in \{1,\dots,N\}$ such that

$$\int_{Q_{i_k}^k} I^k \leq \frac{c}{N}. \tag{5.56}$$

Set $\lambda^k := N^{i_k}$ and define the sequence of truncations as

$$\boldsymbol{e}^k := (\boldsymbol{u}^k - \boldsymbol{u}) \min\left\{1, \frac{\lambda^k}{|\boldsymbol{u}^k - \boldsymbol{u}|}\right\}. \tag{5.57}$$

This sequence satisfies

$$\int_Q |\boldsymbol{e}^k|^2 = \int_{|\boldsymbol{u}^k - \boldsymbol{u}| \leq \lambda^k} |\boldsymbol{u}^k - \boldsymbol{u}|^2 + \int_{|\boldsymbol{u}^k - \boldsymbol{u}| > \lambda^k} |\lambda^k|^2 \leq \int_Q |\boldsymbol{u}^k - \boldsymbol{u}|^2 \xrightarrow[k\to\infty]{} 0,$$

and noting that $\|\boldsymbol{e}^k\|_{L^\infty(Q)} \leq \lambda^k$ we obtain

$$\boldsymbol{e}^k \to 0 \quad \text{strongly in } L^q(\Omega), \text{ for any } q \in [1, \infty). \tag{5.58}$$

In addition, since the derivative is given by

$$\begin{aligned} \nabla \boldsymbol{e}^k &= \nabla(\boldsymbol{u}^k - \boldsymbol{u}) \mathbb{1}_{\{|\boldsymbol{u}^k - \boldsymbol{u}| \leq \lambda^k\}} \\ &\quad + \frac{1}{|\boldsymbol{u}^k - \boldsymbol{u}|}\left(\boldsymbol{I} - \frac{(\boldsymbol{u}^k - \boldsymbol{u}) \otimes (\boldsymbol{u}^k - \boldsymbol{u})}{|\boldsymbol{u}^k - \boldsymbol{u}|}\right) \nabla(\boldsymbol{u}^k - \boldsymbol{u}) \mathbb{1}_{\{|\boldsymbol{u}^k - \boldsymbol{u}| > \lambda^k\}}, \end{aligned} \tag{5.59}$$

we have that $|\nabla \boldsymbol{e}^k| \leq 2|\nabla(\boldsymbol{u}^k - \boldsymbol{u})|$, which implies that

$$\boldsymbol{e}^k \rightharpoonup \boldsymbol{0} \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)^d). \tag{5.60}$$

Another important property is that, by using Chebyshev's inequality, the size of the "bad set" can be estimated:

$$|\{|\boldsymbol{u}^k - \boldsymbol{u}| > \lambda^k\}| \leq |\{|\boldsymbol{u}^k - \boldsymbol{u}| > N\}| \leq \frac{c}{N^2}. \tag{5.61}$$

On the other hand, by approximating with smooth functions it is possible to see that

$$\liminf_{k\to\infty} \int_0^T \langle \partial_t \boldsymbol{u}^k, \boldsymbol{e}^k \rangle \geq 0. \tag{5.62}$$

Testing the momentum equation for $\boldsymbol{u}^k$ with $\boldsymbol{v} = \boldsymbol{e}^k$ then yields

$$\limsup_{k\to\infty} \int_Q (\mathbf{S}^k : \mathbf{D}(\boldsymbol{e}^k) - p_1^k \operatorname{div} \boldsymbol{e}^k) \leq 0. \tag{5.63}$$

Define now the function

$$H^k := (\mathbf{D}(\boldsymbol{u}^k) - \overline{\mathbf{D}}) : (\mathbf{S}^k - \mathbf{S}) \in L^1(Q), \tag{5.64}$$

where $\overline{\mathbf{D}}$ is such that $(\overline{\mathbf{D}}, \mathbf{S}, \theta) \in \mathcal{A}$, with $\overline{\mathbf{D}}$ defined to be equal to $\mathbf{0}$ on the set $\{\mathbf{S} = \mathbf{0}\}$ (this is the set where $\overline{\mathbf{D}}$ is not defined uniquely). Let $\varepsilon > 0$ be arbitrary. In the estimate below we will use the fact that, thanks to (5.61), there is an $N_0 \in \mathbb{N}$ such that for $N \geq N_0$:

$$\int_{\{|\boldsymbol{u}^k-\boldsymbol{u}|>\lambda^k\}} |\mathsf{D}(\boldsymbol{u}) - \overline{\mathsf{D}}|^2 \leq \varepsilon^2. \tag{5.65}$$

Then, taking $N \geq N_0$ and using Young's inequality, we can estimate as follows:

$$\begin{aligned}
\limsup_{k\to\infty} \int_{\{|\boldsymbol{u}^k-\boldsymbol{u}|\leq\lambda^k\}} H^k &\leq \limsup_{k\to\infty} \int_{\{|\boldsymbol{u}^k-\boldsymbol{u}|>\lambda^k\}} (\mathsf{D}(\boldsymbol{u}) - \overline{\mathsf{D}}) : (\mathsf{S}^k - \mathsf{S}) \\
&\quad + \underbrace{\lim_{k\to\infty} \left| \int_Q (\mathsf{D}(\boldsymbol{u}) - \overline{\mathsf{D}}) : (\mathsf{S}^k - \mathsf{S}) \right|}_{=0} + \limsup_{k\to\infty} \int_Q (\mathsf{S}^k - \mathsf{S}) : \mathsf{D}(\boldsymbol{e}^k) \\
&\quad + \limsup_{k\to\infty} \int_{\{|\boldsymbol{u}^k-\boldsymbol{u}|>\lambda^k\}} (\mathsf{S}^k - \mathsf{S}) : \mathsf{D}(\boldsymbol{e}^k) \\
&\overset{(5.63)}{\underset{(5.59)}{\leq}} c \underbrace{\lim_{k\to\infty} \left(\int_{\{|\boldsymbol{u}^k-\boldsymbol{u}|>\lambda^k\}} |\mathsf{D}(\boldsymbol{u}) - \overline{\mathsf{D}}|^2 \right)^{1/2}}_{\leq\varepsilon} + \limsup_{k\to\infty} \int_Q p_1^k \operatorname{div} \boldsymbol{e}^k \\
&\quad + c \limsup_{k\to\infty} \int_{\{|\boldsymbol{u}^k-\boldsymbol{u}|>\lambda^k\}} \frac{1}{|\boldsymbol{u}^k - \boldsymbol{u}|} |\mathsf{S}^k - \mathsf{S}| |\nabla \boldsymbol{u}^k - \nabla \boldsymbol{u}| \\
&\overset{(5.59)}{\leq} c \left(\varepsilon + \limsup_{k\to\infty} \int_{\{|\boldsymbol{u}^k-\boldsymbol{u}|>\lambda^k\}} \frac{\lambda^k}{|\boldsymbol{u}^k - \boldsymbol{u}|} I^k \right) \\
&= c \left(\varepsilon + \limsup_{k\to\infty} \left(\int_{\{N^{i_k+1}\geq|\boldsymbol{u}^k-\boldsymbol{u}|>N^{i_k}\}} \frac{N^{i_k}}{|\boldsymbol{u}^k - \boldsymbol{u}|} I^k \right.\right. \\
&\quad \left.\left. + \int_{\{|\boldsymbol{u}^k-\boldsymbol{u}|>N^{i_k+1}\}} \frac{N^{i_k}}{|\boldsymbol{u}^k - \boldsymbol{u}|} I^k \right) \right) \\
&\leq c \left(\varepsilon + \int_{Q^k_{i_k}} I^k + \frac{1}{N^{i_k}} \int_Q I^k \right) \\
&\overset{(5.47)}{\underset{(5.56)}{\leq}} c(\varepsilon + \frac{1}{N}),
\end{aligned}$$

and since ε is arbitrary, this implies that

$$\limsup_{k\to\infty} \int_{|\boldsymbol{u}^k-\boldsymbol{u}|\leq\lambda^k} H^k \leq \frac{c}{N}. \tag{5.66}$$

On the other hand, by the monotonicity of $\boldsymbol{\mathcal{D}}^k$ we can write

$$\begin{aligned}
H^k &= (\mathsf{D}(\boldsymbol{u}^k) - \boldsymbol{\mathcal{D}}^k(\mathsf{S}, \theta^k)) : (\mathsf{S}^k - \mathsf{S}) + (\boldsymbol{\mathcal{D}}^k(\mathsf{S}, \theta^k) - \overline{\mathsf{D}}) : (\mathsf{S}^k - \mathsf{S}) \\
&\geq (\boldsymbol{\mathcal{D}}^k(\mathsf{S}, \theta^k) - \overline{\mathsf{D}}) : (\mathsf{S}^k - \mathsf{S}),
\end{aligned}$$

and since $\boldsymbol{\mathcal{D}}^k(\mathsf{S}, \theta^k)$ converges pointwise almost everywhere to $\overline{\mathsf{D}}$, with the help of the dominated convergence theorem (compare with (5.36)) we see that the right hand side in the above inequality converges to zero strongly in $L^1(Q)$, or in other words we have that

$$(H^k)^- \to 0 \quad \text{strongly in } L^1(Q), \tag{5.67}$$

which together with (5.66) implies that

$$\limsup_{k\to\infty} \int_{\{|\boldsymbol{u}^k-\boldsymbol{u}|\le\lambda^k\}} |H^k| \le \frac{c}{N}. \tag{5.68}$$

Now, for an arbitrary $\eta \in (0,1)$ Hölder's and Chebyshev's inequalities yield

$$\begin{aligned}\int_Q |H^k|^\eta &\le |Q|^{1-\eta}\left(\int_{\{|\boldsymbol{u}^k-\boldsymbol{u}|\le\lambda^k\}} |H^k|\right)^\eta + |\{|\boldsymbol{u}^k-\boldsymbol{u}|>N\}|^{1-\eta}\left(\int_Q |H^k|\right)^\eta \\ &\overset{(5.47)}{\le} |Q|^{1-\eta}\left(\int_{\{|\boldsymbol{u}^k-\boldsymbol{u}|\le\lambda^k\}} |H^k|\right)^\eta + c\left(\frac{1}{N^2}\right)^{1-\eta}.\end{aligned}$$

Taking lim sup and using (5.68) we conclude that, possibly up to a subsequence, $H^k \to 0$ almost everywhere. Since H^k is bounded uniformly in $L^1(Q)$ thanks to (5.38), Chacon's biting lemma and Vitali's theorem then guarantee the existence of a nonincreasing sequence of sets $E_j \subset Q$ such that $|E_j| \to 0$ as $j \to \infty$ and

$$H^k \to 0 \quad \text{strongly in } L^1(Q \setminus E_j), \text{ for any } j \in \mathbb{N}.$$

Recalling the convergence properties of $\mathbf{S}^k$ this can be reformulated as

$$\limsup_{k\to\infty} \int_{Q\setminus E_j} \mathbf{S}^k : \mathbf{D}^k = \int_{Q\setminus E_j} \mathbf{S} : \mathbf{D}, \text{ for any } j \in \mathbb{N}. \tag{5.69}$$

Lemma 5.1.2 then implies that $(\mathbf{D}, \mathbf{S}, \theta) \in \mathcal{A}$ almost everywhere in $Q \setminus E_j$ and that $\mathbf{S}^k : \mathbf{D}^k \to \mathbf{S} : \mathbf{D}$ weakly in $L^1(Q \setminus E_j)$, for any $j \in \mathbb{N}$. Since the measure of the sets E_j tends to zero, the constitutive relation can be identified almost everywhere:

$$(\mathbf{D}, \mathbf{S}, \theta) \in \mathcal{A} \quad \text{a.e. in } Q. \tag{5.70}$$

As for the entropy inequality, first note that for a given non-negative function $\psi \in C([0,T]; W^{1,\infty}(\Omega))$ and $\varepsilon > 0$ there is a $\delta > 0$ such that for any measurable set $E \subset Q$ with $|E| \le \delta$:

$$\int_E \mathbf{S} : \mathbf{D}\psi \le \varepsilon.$$

Then, choosing $j \in \mathbb{N}$ large enough so that $|E_j| \le \delta$, we have

$$\liminf_{k\to\infty} \int_Q \mathbf{S}^k : \mathbf{D}^k\psi \ge \liminf_{k\to\infty} \int_{Q\setminus E_j} \mathbf{S}^k : \mathbf{D}^k\psi = \int_{Q\setminus E_j} \mathbf{S} : \mathbf{D}\psi \ge \int_Q \mathbf{S} : \mathbf{D}\psi - \varepsilon. \tag{5.71}$$

Taking into account the convergences (5.49) we can therefore conclude that

$$\langle \partial_t \theta, \psi \rangle + \int_\Omega (-\theta \boldsymbol{u} + \kappa(\theta)\nabla\theta) \cdot \nabla \boldsymbol{u} \geq \int_\Omega \mathbf{S} : \mathbf{D}(\boldsymbol{u})\psi \quad \forall\, \psi \in W^{1,\infty}(\Omega), \psi \geq 0,$$

in the sense of measures.

The attainment of the initial conditions can be proved in exactly the same way as in [MZ18]. The argument for the temperature is based on the inequality

$$\int_\Omega \sqrt{\theta^k(t,\cdot)}\psi - \int_\Omega \sqrt{\theta_0}\psi + \int_0^t \int_\Omega \left(-\sqrt{\theta^k}\boldsymbol{u}^k + \frac{\kappa(\theta^k)\nabla\theta^k}{2\sqrt{\theta^k}} \right) \cdot \nabla\psi \geq 0,$$

which holds for any positive $\psi \in W^{1,\infty}(\Omega)$. This inequality can be obtained at a level of sufficiently regular approximations (say, for $\theta^{k,n,l}$) and is stable under passage to the limit. In summary, we have proved the following result.

Theorem 5.1.5. *Suppose $\Omega \subset \mathbb{R}^d$ is a convex polyhedron satisfying the angle condition (5.42) if $d = 3$ or any convex polygon if $d = 2$, and let $\{\Sigma^n, V^n, M^n, U^n\}_{n\in\mathbb{N}}$ be a family of finite element spaces satisfying Assumptions 2.5.1–2.5.5. Then, for $k, n, m, l \in \mathbb{N}$, there exists a sequence of solutions $\{\mathbf{S}_j^{k,n,m,l}, \boldsymbol{u}_j^{k,n,m,l}, p_j^{k,n,m,l}, \theta_j^{k,n,m,l}\}_{j=1}^{T/\tau_m}$ of Formulation $\tilde{\mathrm{A}}_{\mathrm{k,n,m,l}}$, and there exist functions*

$$\begin{aligned} \mathbf{S} &\in L^2_{\mathrm{sym,tr}}(Q)^{d\times d}, \\ \boldsymbol{u} &\in L^2(0,T; W^{1,2}_{\boldsymbol{n},\mathrm{div}}(\Omega)^d) \cap L^\infty(0,T; L^2_{\mathrm{div}}(\Omega)^d), \\ p &\in L^{\check{2}}(Q), \\ \theta &\in L^n(0,T; W^{1,n}(\Omega)) \cap L^\infty(0,T; L^1(\Omega)) \text{ for } n \in [1, \tfrac{5}{4}), \end{aligned}$$

such that the corresponding time interpolants $\overline{\mathbf{S}}^{k,n,m,l}$, $\overline{\boldsymbol{u}}^{k,n,m,l}$, $\tilde{\boldsymbol{u}}^{k,n,m,l}$, $\overline{p}^{k,n,m,l}$, $\overline{\theta}^{k,n,m,l}$ satisfy (up to subsequences):

$$\begin{aligned} \overline{\mathbf{S}}^{k,n,m,l} &\rightharpoonup \mathbf{S} && \text{weakly in } L^2(Q)^{d\times d}, \\ \overline{\boldsymbol{u}}^{k,n,m,l} &\rightharpoonup \boldsymbol{u} && \text{weakly in } L^2(0,T; W^{1,2}_{\boldsymbol{n}}(\Omega)^d), \\ \tilde{\boldsymbol{u}}^{k,n,m,l}, \overline{\boldsymbol{u}}^{k,n,m,l} &\overset{*}{\rightharpoonup} \boldsymbol{u} && \text{weakly* in } L^\infty(0,T; L^2(\Omega)^d), \\ \overline{p}^{k,n,m,l} &\rightharpoonup p && \text{weakly in } L^{\check{2}}(Q), \\ \overline{\theta}^{k,n,m,l} &\rightharpoonup \theta && \text{weakly in } L^n(0,T; W^{1,n}(\Omega)), \text{ for } n \in [1, \tfrac{5}{4}), \end{aligned}$$

and $(\mathbf{S}, \boldsymbol{u}, p, \theta)$ solves Formulation $\tilde{\mathrm{A}}$, with the limits taken in the order $m \to \infty$, $n \to \infty$, $l \to \infty$, and $k \to \infty$.

From the proof of Theorem 5.1.5 we see that the only bottleneck that prevents us from obtaining a convergence result for more general r-graphs is that we are missing the corresponding localised Minty lemma (Lemma 5.1.2 is tailored to a specific 2-graph). Suppose for instance that the constitutive relation is defined by an explicit continuous function $\boldsymbol{\mathcal{D}} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ satisfying for some $r > \frac{3d}{d+2}$:

(E1) $\boldsymbol{\mathcal{D}}(\mathbf{0}, s) = \mathbf{0}$ for any $s \in \mathbb{R}$;

(E2) For every $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$,

$$(\boldsymbol{\mathcal{D}}(\boldsymbol{\sigma}_1, s) - \boldsymbol{\mathcal{D}}(\boldsymbol{\sigma}_2, s)) : (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \geq 0 \text{ for fixed } s \in \mathbb{R};$$

(E3) There is a non-negative function $m \in L^1(Q)$ and a constant $c > 0$ such that

$$\boldsymbol{\mathcal{D}}(\boldsymbol{\sigma}, s) : \boldsymbol{\sigma} \geq -m + c(|\boldsymbol{\mathcal{D}}(\boldsymbol{\sigma}, s)|^r + |\boldsymbol{\sigma}|^{r'}) \quad \text{for all } \boldsymbol{\sigma} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, s \in \mathbb{R};$$

(E4) There is a constant $\tilde{c} > 0$ such that for any $\boldsymbol{\sigma} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $s \in \mathbb{R}$:

$$|\boldsymbol{\mathcal{D}}(\boldsymbol{\sigma}, s)| \leq \tilde{c}(|\boldsymbol{\sigma}|^{r'-1} + 1);$$

(E5) For fixed $s \in \mathbb{R}$ we have that $\mathrm{tr}(\boldsymbol{\mathcal{D}}(\boldsymbol{\sigma}, s)) = 0$ if and only if $\mathrm{tr}(\boldsymbol{\sigma}) = 0$, for any $\boldsymbol{\sigma} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$;

Then a convergence result analogous to Theorem 5.1.5 will hold, with the solution belonging to

$$\begin{gathered}\mathsf{S} \in L^{r'}_{\mathrm{sym,tr}}(Q)^{d\times d},\\ \boldsymbol{u} \ \in L^r(0,T; W^{1,r}_{\boldsymbol{n},\mathrm{div}}(\Omega)^d) \cap L^\infty(0,T; L^2_{\mathrm{div}}(\Omega)^d),\\ p \in L^{\check{r}}(Q),\\ \theta \in L^n(0,T; W^{1,n}(\Omega)) \cap L^\infty(0,T; L^1(\Omega)) \text{ for } n \in [1, \tfrac{5}{4}).\end{gathered}$$

Observe that the condition $r > \frac{3d}{d+2}$ ensures that all the terms in the equation for the total energy E are precompact and so passage to the limit is possible. In addition, the condition $r > 2\frac{d+1}{d+2}$ is less restrictive than $r > \frac{3d}{d+2}$ which means that the same L^∞-truncation argument can be applied in this case.

An example of a constitutive relation satisfying (E1)–(E4) is given by

$$\boldsymbol{\mathcal{D}}(\mathsf{S}, \theta) = \frac{1}{2\mu(\theta)} \frac{(|\mathsf{S}| - \tau(\theta))^+}{|\mathsf{S}|} \left(\frac{|\mathsf{S}|}{2\mu(\theta)}\right)^{r'-2} \mathsf{S}, \tag{5.72}$$

where $\mu, \tau : \mathbb{R} \to \mathbb{R}$ are continuous functions that satisfy

$$c_1 \leq \mu(s), \tau(s) \leq c_2 \text{ for any } s \in \mathbb{R}, \tag{5.73}$$

for two positive constants c_1, c_2, assuming $r \leq 2$; this condition is used to prove (E4), using that $|\mathbf{S}|^{r'-2} \leq |\mathbf{S}|^{r'-1} + 1$ if $r' \geq 2$; if $\tau \equiv 0$ then (E4) is true for any $r > 1$. The relation (5.72) describes a Herschel–Bulkley type fluid with a temperature dependent viscosity and yield stress.

Remark 5.1.6. *The form of the convective term $\mathcal{B}_k$ employed in Formulation $\tilde{\mathrm{A}}_{\mathrm{k,n,m,l}}$ is not entirely satisfactory because it involves performing a mollification and computing a Helmholtz decomposition, which could prove challenging in practice. However, we expect that the convergence result holds for the formulation employing the usual skew-symmetric form of the convective term, assuming that the Fortin operator Π^n_V is quasi-local, which would allow one to perform the L^∞-truncation argument at the discrete level.*

Remark 5.1.7. *The quasi-compressibility approximation is a technical tool that does not seem to be necessary in practice; i.e. when performing simulations it is possible to work with the usual divergence-free constraint, assuming the velocity and pressure pair is inf-sup stable. This approximation step could be avoided in the proof if the time and space discretisation limits are taken simultaneously. In that case, if one denotes by $h^{k,n,l}_j$ the solution of the analogue of the Neumann problem (5.39) corresponding to $p^{k,n,l}_j$, for $j \in \{1, \ldots, T/\tau_n\}$, one could test the discrete momentum equation with $\Pi^n_V \nabla h^{k,n,l}_j$ and obtain the uniform estimate for the pressure:*

$$\sum_{j=1}^{T/\tau_m} \tau_m \|p^{k,n,l}_j\|^{\check{2}}_{L^{\check{2}}(\Omega)} \leq c. \tag{5.74}$$

The bound on the term involving the time derivative is the only one worth mentioning, as the others can be handled similarly. Suppose for the sake of simplicity that $V^n_{\mathrm{div}} \subset W^{1,2}_{\boldsymbol{n},\mathrm{div}}(\Omega)^d$. Then, assuming that the Fortin projector has optimal approximation properties, the regularity estimate (5.40) and the Hölder and Young inequalities imply that

$$\begin{aligned}\left|\frac{1}{\tau_n}\int_\Omega (\boldsymbol{u}^{k,n,l}_j - \boldsymbol{u}^{k,n,l}_{j-1}) \cdot \Pi^n \nabla h^{k,n,l}_j\right| &= \left|\frac{1}{\tau_n}\int_\Omega (\boldsymbol{u}^{k,n,l}_j - \boldsymbol{u}^{k,n,l}_{j-1}) \cdot (\Pi^n \nabla h^{k,n,l}_j - \nabla h^{k,n,l}_j)\right| \\ &\leq c(\eta)\frac{h_n^2}{\tau_n^2}\|\boldsymbol{u}^{k,n,l}_j - \boldsymbol{u}^{k,n,l}_{j-1}\|^2_{L^2(\Omega)} + \eta\|p^{k,n,l}_j\|^{\check{2}}_{L^{\check{2}}(\Omega)}.\end{aligned}$$

Multiplying by τ_n and summing over $j \in \{1, \dots, T/\tau_n\}$ yields (5.74), if one assumes that $h_n^2 = O(\tau_n)$. This argument could be applied, for instance, to the isothermal system with Navier's slip boundary conditions (compare with Theorem 3.1.5, where the pressure was not included in the analysis). In addition, we expect that this argument could be applied to the anisothermal system, if one assumes the mesh is such that the discrete maximum principle is satisfied; this would ensure the positivity of the discrete temperatures, and in turn that the analogue of estimate (5.30) holds, allowing one to take the limits corresponding to n and m simultaneously.

5.2 Steady buoyancy-driven flow

In this section we will analyse a system describing the steady state of a buoyancy-driven heat-conducting fluid; while remaining of great importance in applications, this setting will allow us to avoid many of the technical complications that arose in the previous section. Namely, unlike Formulation $\tilde{\mathrm{A}}_{\mathrm{k,n,m,l}}$, the finite element formulation considered in this section will employ neither a form of the convective constructed via mollification and the Helmholtz decomposition, nor the quasi-compressibility approximation. Moreover, the convergence argument will only involve taking a single limit $n \to \infty$. In addition, we will introduce a preconditioner for this system that follows the ideas presented in Chapter 4.

The steady form of the Oberbeck–Boussinesq [Obe79, Bou03] approximation used in the modelling of natural convection reads (c.f. (1.24)):

$$
\begin{aligned}
-\operatorname{div} \mathbf{S} + \rho_* \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p &= -\rho_* \beta g(\theta - \theta_C)\boldsymbol{e}_d && \text{in } \Omega, && \text{(5.75a)}\\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, && \text{(5.75b)}\\
-\operatorname{div}(\kappa(\theta)\nabla\theta) + \rho_* c_p \operatorname{div}(\boldsymbol{u}\theta) + \beta\rho_* g\theta \boldsymbol{u}\cdot\boldsymbol{e}_d &= \mathbf{S} : \mathbf{D}(\boldsymbol{u}) && \text{in } \Omega, && \text{(5.75c)}
\end{aligned}
$$

where ρ_* and θ_C are reference values for the density and temperature, g is the acceleration due to gravity, β is the volumetric coefficient of thermal expansion, c_p is the specific heat capacity at constant pressure, and $\boldsymbol{e}_d$ is the unit vector pointing against gravity. The system is supplemented with the boundary conditions

$$
\boldsymbol{u}|_{\partial\Omega} = \mathbf{0}, \qquad \theta|_{\Gamma_D} = \theta_b, \qquad \kappa(\theta)\nabla\theta\cdot\boldsymbol{n}|_{\partial\Omega\setminus\Gamma_D} = 0, \tag{5.76}
$$

where Γ_D is a relatively open subset of $\partial\Omega$ with $|\Gamma_D| \neq 0$, $\boldsymbol{n}$ is the unit outward-pointing normal vector to the boundary, and $\theta_b \in H^{1/2}_{00}(\Gamma_D) := W^{1/2,2}_{00}(\Gamma_D)$ is a given temperature distribution on Γ_D. The system will be closed with the implicit constitutive relation (5.5).

In many applications the effects of viscous dissipation are ignored, i.e. only the first two terms in the temperature equation (5.75c) are kept. However, it has been observed that in some cases the effects of the viscous dissipation term $\mathbf{S}:\mathbf{D}(\boldsymbol{u})$ are non-negligible and should be taken into account [HMW75, THTS74, VPC76, Ost58]. Furthermore, as noted in [BLP92, THTS74], the viscous dissipation must be balanced with the adiabatic heating term $\beta\rho_* g\theta\boldsymbol{u}\cdot\boldsymbol{e}_d$; for a mathematically rigorous derivation of the system (5.75) see [KRT00]. The existence of distributional solutions of (5.75) with non-Newtonian rheology of power-law type was shown in [Rou01, NR01].

We will introduce a finite element approximation of the system (5.75) and prove convergence of the sequence of finite element approximations to a weak solution. For the sake of simplicity, we will neglect the viscous dissipation in the convergence analysis. However, this can be included in the numerical algorithm without any difficulties. As seen in the previous section, the main challenge associated with this term in the analysis stems from the fact that $\mathbf{S}:\mathbf{D}(\boldsymbol{u})$ belongs a priori to $L^1(\Omega)$ only, and hence a suitable notion of renormalised solution must be employed for the temperature equation. We would expect that by imposing certain restrictions on the mesh, and for $\mathbb{P}_1$ elements, a similar convergence result would hold for an appropriately defined renormalised solution (c.f. [CDCRG$^+$07]). When restricted to constant rheological parameters and the isothermal problem, the convergence result here improves on the result for r-graphs from [DKS13] by extending it to cover the whole admissible range $r > \frac{2d}{d+2}$, even without pointwise divergence-free elements. This is possible by making use of reconstruction operators, which in recent years were introduced to restore the pressure-robustness in finite element formulations (see e.g. [JLM$^+$17]).

Let $\mathcal{B}$ be defined as the usual skew-symmetric form of the convective term:

$$\mathcal{B}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) := \begin{cases} -\displaystyle\int_\Omega \boldsymbol{u}\otimes\boldsymbol{v}:\nabla\boldsymbol{w}, & \text{if } V^n_{\text{div}} \subset W^{1,1}_{0,\text{div}}(\Omega)^d, \\ \dfrac{1}{2}\displaystyle\int_\Omega \boldsymbol{u}\otimes\boldsymbol{w}:\nabla\boldsymbol{v} - \boldsymbol{u}\otimes\boldsymbol{v}:\nabla\boldsymbol{w}, & \text{otherwise.} \end{cases}$$

As seen in Chapter 3, this trilinear form satisfies $\mathcal{B}(\boldsymbol{v},\boldsymbol{v},\boldsymbol{v}) = 0$ for any $\boldsymbol{v}\in W^{1,\infty}_0(\Omega)^d$, regardless of whether $\boldsymbol{v}$ is divergence-free or not, and it reduces to the original trilinear form $-\int_\Omega(\boldsymbol{u}\otimes\boldsymbol{v}):\nabla\boldsymbol{w}$ if $\operatorname{div}\boldsymbol{v} = 0$.

Let us now define

$$\tilde{r} := \min\{r', r^*/2\}, \text{ where } r^* := \begin{cases} \frac{dr}{d-r} & \text{if } r < d, \\ \infty, & \text{otherwise.} \end{cases}$$

Observe that the condition $\tilde{r} > 1$ is equivalent to $r > \frac{2d}{d+2}$, which is the natural condition required to have a well-defined weak form of the convective term, because it ensures that $W^{1,r}(\Omega)^d \hookrightarrow L^2(\Omega)^d$. In this case, for exactly divergence-free functions $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V^n_{\mathrm{div}}$ one has that

$$|\mathcal{B}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq \int_\Omega |\boldsymbol{u} \otimes \boldsymbol{v} : \nabla \boldsymbol{w}| \leq c\|\boldsymbol{u}\|_{W^{1,r}(\Omega)} \|\boldsymbol{v}\|_{W^{1,r}(\Omega)} \|\boldsymbol{w}\|_{W^{1,\tilde{r}'}(\Omega)}. \tag{5.77}$$

Otherwise one needs the stronger assumption $r > \frac{2d}{d+1}$; this ensures that there is an $s \in (1, \infty)$ such that $\frac{1}{r} + \frac{1}{2\tilde{r}} + \frac{1}{s} = 1$ and so (c.f. [DKS13])

$$\begin{aligned} \int_\Omega |\boldsymbol{u} \otimes \boldsymbol{w} : \nabla \boldsymbol{v}| &\leq \|\boldsymbol{u}\|_{L^{2\tilde{r}}(\Omega)} \|\boldsymbol{v}\|_{W^{1,r}(\Omega)} \|\boldsymbol{w}\|_{L^s(\Omega)} \\ &\leq c\|\boldsymbol{u}\|_{W^{1,r}(\Omega)} \|\boldsymbol{v}\|_{W^{1,r}(\Omega)} \|\boldsymbol{w}\|_{W^{1,\tilde{r}'}(\Omega)}, \end{aligned} \tag{5.78}$$

for any $\boldsymbol{u}, \boldsymbol{v} \in W^{1,r}(\Omega)^d, \boldsymbol{w} \in W^{1,\tilde{r}'}(\Omega)^d$. Thus we deduce that the trilinear form $\mathcal{B}(\cdot,\cdot,\cdot)$ is bounded on $W^{1,r}(\Omega)^d \times W^{1,r}(\Omega)^d \times W^{1,\tilde{r}'}(\Omega)^d$ if $r > \frac{2d}{d+2}$ when using exactly divergence-free elements and if $r > \frac{2d}{d+1}$ otherwise. This does not pose a problem when working with the constitutive relation (5.5) (for which $r = 2$), but for relations with more general r-growth the more demanding requirement that $r > \frac{2d}{d+1}$ would impose a restriction on the convergence result that can be obtained (see [DKS13, Thm. 18]). In order to circumvent this issue we shall make use of a reconstruction operator.

Assumption 5.2.1 (Reconstruction operator π^n)**.** *Let X^n be an auxiliary $H(\mathrm{div};\Omega)$-conforming finite element space. There exists a map $\pi^n \colon W^{1,1}(\Omega)^d \to V^n + X^n$ (usually called a reconstruction operator) that satisfies:*

- *(Preservation of Divergence). If $\boldsymbol{v} \in V^n_{\mathrm{div}}$ then $\mathrm{div}(\pi^n \boldsymbol{v}) = 0$ pointwise.*
- *(Consistency). For every $\boldsymbol{v} \in V^n$ and $K \in \mathcal{T}_n$ it holds that*

$$\|\boldsymbol{v} - \pi^n \boldsymbol{v}\|_{L^s(K)} \leq c h_K^m |\boldsymbol{v}|_{W^{m,s}(K)}, \qquad \textit{for } s \in [1, \infty),\ m \in \{0, 1, 2\}.$$

Operators with the properties described above have been constructed in [Lin14, LM16, LMT16, LMW17, JLM+17] for elements with discontinuous pressures; the construction is based on the interpolation operators associated with the Raviart–Thomas and Brezzi–Douglas–Marini elements. A slightly more complicated construction for elements with continuous pressures was introduced in [LLM17]. These reconstruction operators have been employed to obtain pressure-robust discretisations by "repairing" the L^2-orthogonality between discretely divergence-free functions and gradient fields; see [JLM+17] for more details. In order to exploit the advantages of this framework one has to replace the L^2 inner products in the discrete formulation in the following way:

$$\int_\Omega \boldsymbol{w} \cdot \boldsymbol{v} \mapsto \int_\Omega \boldsymbol{w} \cdot \pi^n \boldsymbol{v}, \tag{5.79}$$

where $\boldsymbol{v} \in V^n$ is a test function. As for the convective term, let us define

$$\tilde{\mathcal{B}}_n(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := \begin{cases} -\displaystyle\int_\Omega \boldsymbol{u} \otimes \boldsymbol{v} : \nabla \boldsymbol{w}, & \text{if } V^n_{\mathrm{div}} \subset W^{1,1}_{0,\mathrm{div}}(\Omega)^d, \\ -\int_\Omega \boldsymbol{u} \otimes \pi^n \boldsymbol{v} : \nabla \boldsymbol{w}, & \text{otherwise.} \end{cases} \tag{5.80}$$

From the properties of π^n stated in Assumption 5.2.1 one readily sees that the trilinear form $\tilde{\mathcal{B}}_n$ is bounded on $W^{1,r}(\Omega)^d \times W^{1,r}(\Omega)^d \times W^{1,\tilde{r}'}(\Omega)^d$, and that $\tilde{\mathcal{B}}_n(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v}) = 0$ for any $\boldsymbol{v} \in V^n_{\mathrm{div}}$.

For the advective term for the temperature one can analogously define the trilinear form

$$\mathcal{C}(\boldsymbol{u}, \theta, \eta) := \begin{cases} -\displaystyle\int_\Omega \boldsymbol{u}\theta \cdot \nabla \eta, & \text{if } V^n_{\mathrm{div}} \subset W^{1,1}_{0,\mathrm{div}}(\Omega)^d, \\ \dfrac{1}{2} \displaystyle\int_\Omega \boldsymbol{u}\eta \cdot \nabla \theta - \boldsymbol{u}\theta \cdot \nabla \eta, & \text{otherwise,} \end{cases}$$

which is well defined and bounded on $W^{1,r}(\Omega)^d \times H^1(\Omega) \times W^{1,\infty}(\Omega)$ assuming that $r > \frac{2d}{d+2}$. In addition, this form satisfies $\mathcal{C}(\boldsymbol{u}, \eta, \eta) = 0$ for any $\eta \in W^{1,\infty}(\Omega)$, regardless of whether $\boldsymbol{u}$ is divergence-free or not. The form $\mathcal{C}$ does not impose additional restrictions like $\mathcal{B}$ does for small r, but a trilinear form using a reconstruction operator $\tilde{\mathcal{C}}_n$ could be used instead (and defined analogously).

5.2.1 Finite Element Approximation

Let us now set the physical constants to unity for ease of readability (appropriate non-dimensional forms of the system will be employed in Section 5.3). Let $\hat{\theta}_b \in H^1(\Omega)$ be such that $\hat{\theta}_b|_{\Gamma_D} = \theta_b$ (such a function exists because $\theta_b \in H^{1/2}_{00}(\Gamma_D)$). We can now define the weak formulation of the system (without viscous heating).

Formulation $\mathbf{A_0}$. Find $(\mathsf{S}, \theta, \boldsymbol{u}, p) \in L^2_{\mathrm{sym,tr}}(\Omega)^{d\times d} \times (\hat{\theta}_b + H^1_{\Gamma_D}(\Omega)) \times H^1_0(\Omega)^d \times L^2_0(\Omega)$ such that:

$$\int_\Omega \mathsf{S} : \mathsf{D}(\boldsymbol{v}) - \int_\Omega \boldsymbol{u} \otimes \boldsymbol{u} : \mathsf{D}(\boldsymbol{v}) - \int_\Omega p \operatorname{div} \boldsymbol{v} = \int_\Omega \theta \boldsymbol{v} \cdot \boldsymbol{e}_d \quad \forall \boldsymbol{v} \in C^\infty_0(\Omega)^d, \tag{5.81a}$$

$$-\int_\Omega q \operatorname{div} \boldsymbol{u} = 0 \quad \forall q \in C^\infty_0(\Omega), \tag{5.81b}$$

$$\int_\Omega \kappa(\theta) \nabla \theta \cdot \nabla \eta - \boldsymbol{u}\theta \cdot \nabla \eta = 0 \quad \forall \eta \in C^\infty_{\Gamma_D}(\Omega), \tag{5.81c}$$

$$(\mathsf{D}(\boldsymbol{u}), \mathsf{S}, \theta) \in \mathcal{A} \quad \text{a.e. in } \Omega. \tag{5.81d}$$

Let $\hat{\theta}_b^n$ be the standard Scott–Zhang interpolant of $\hat{\theta}_b$ into $\hat{U}^n$, where $\hat{U}^n$ is the same finite element space as U^n, but without strongly imposed boundary conditions. The discrete formulations will employ the continuous explicit approximations of the graph $\boldsymbol{\mathcal{S}}^n$ and $\boldsymbol{\mathcal{D}}^n$, defined in (5.10). We have everything in place to state the finite element approximation of the problem.

Formulation $\mathbf{A_0^n}$. Find $(\theta^n, \boldsymbol{u}^n, p^n) \in (\hat{\theta}_b^n + U^n) \times V^n \times M_0^n$ such that:

$$\begin{aligned}
\int_\Omega \boldsymbol{\mathcal{S}}^n(\mathbf{D}(\boldsymbol{u}^n), \theta^n) : \mathbf{D}(\boldsymbol{v}) + \mathcal{B}(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v}) - \int_\Omega p^n \operatorname{div} \boldsymbol{v} = \int_\Omega \theta^n \boldsymbol{v} \cdot \boldsymbol{e}_d \qquad & \forall\, \boldsymbol{v} \in V^n, \\
-\int_\Omega q \operatorname{div} \boldsymbol{u}^n = 0 \qquad & \forall\, q \in M^n, \\
\int_\Omega \kappa(\theta^n) \nabla(\theta^n) \cdot \nabla \eta + \mathcal{C}(\boldsymbol{u}^n, \theta^n, \eta) = 0 \qquad & \forall\, \eta \in U^n.
\end{aligned}$$

In case one wishes to compute the shear stress directly, a 4-field formulation may be employed instead. We refer to this formulation as Formulation B_0^n. We will prove that the solutions to the discrete formulations A_0^n and B_0^n converge to a weak solution of Formulation A_0.

Formulation $\mathbf{B_0^n}$. Find $(\mathbf{S}^n, \theta^n, \boldsymbol{u}^n, p^n) \in \Sigma^n \times (\hat{\theta}_b^n + U^n) \times V^n \times M_0^n$ such that:

$$\int_\Omega (\boldsymbol{\mathcal{D}}^n(\mathbf{S}^n, \theta^n) - \mathbf{D}(\boldsymbol{u}^n)) : \boldsymbol{\tau} = 0 \qquad \forall\, \boldsymbol{\tau} \in \Sigma^n, \tag{5.82a}$$

$$\int_\Omega \mathbf{S}^n : \mathbf{D}(\boldsymbol{v}) + \mathcal{B}(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v}) - \int_\Omega p^n \operatorname{div} \boldsymbol{v} = \int_\Omega \theta^n \boldsymbol{v} \cdot \boldsymbol{e}_d \quad \forall\, \boldsymbol{v} \in V^n, \tag{5.82b}$$

$$-\int_\Omega q \operatorname{div} \boldsymbol{u}^n = 0 \qquad \forall\, q \in M^n, \tag{5.82c}$$

$$\int_\Omega \kappa(\theta^n) \nabla \theta^n \cdot \nabla \eta + \mathcal{C}(\boldsymbol{u}^n, \theta^n) = 0 \qquad \forall\, \eta \in U^n. \tag{5.82d}$$

We define Formulations $\tilde{\mathrm{A}}_0^n$ and $\tilde{\mathrm{B}}_0^n$ as the analogues of the formulations A_0^n and B_0^n, respectively, in which we replace $\mathcal{B}$ and $\mathcal{C}$ by $\tilde{\mathcal{B}}_n$ and $\tilde{\mathcal{C}}_n$. The following lemma asserts that all of these formulations have a solution.

Lemma 5.2.2. *Suppose the material parameters satisfy condition* (5.6) *and suppose that $\{U^n, V^n, M^n\}_{n\in\mathbb{N}}$ (respectively $\{\Sigma^n, U^n, V^n, M^n\}_{n\in\mathbb{N}}$) is a family of finite element spaces satisfying Assumptions 2.5.1 and 2.5.3–2.5.5 (resp. 2.5.1–2.5.5). In the case of formulations $\tilde{\mathrm{A}}_0^n$ and $\tilde{\mathrm{B}}_0^n$ suppose further that Assumption 5.2.1 holds. Then, for every $n \in \mathbb{N}$, Formulations A_0^n and $\tilde{\mathrm{A}}_0^n$ (resp. B_0^n and $\tilde{\mathrm{B}}_0^n$) admit a solution $(\theta^n, \boldsymbol{u}^n, p^n) \in (\hat{\theta}_b^n + U^n) \times V^n \times M_0^n$ (resp. $(\mathbf{S}^n, \theta^n, \boldsymbol{u}^n, p^n) \in \Sigma^n \times (\hat{\theta}_b^n + U^n) \times V^n \times M_0^n$). Moreover, the following a priori estimate holds:*

$$\|\boldsymbol{u}^n\|_{H^1(\Omega)} + \|\theta^n\|_{H^1(\Omega)} + \|p^n\|_{L^2(\Omega)} + \|\mathbf{S}^n\|_{L^2(\Omega)} \le c, \tag{5.83a}$$

where the constant c is independent of n; we denote $\mathbf{S}^n := \boldsymbol{\mathcal{S}}^n(\mathbf{D}(\boldsymbol{u}^n), \theta^n)$ in the case of Formulations A_0^n and $\tilde{\mathrm{A}}_0^n$. In addition, for Formulations B_0^n and $\tilde{\mathrm{B}}_0^n$ we have

$$\|\boldsymbol{\mathcal{D}}^n(\mathbf{S}^n, \theta^n)\|_{L^2(\Omega)} \leq c. \tag{5.83b}$$

Proof. We will carry out the proof for Formulation B_0^n; the proof for the other formulations is analogous with some simplifications. The existence proof will make use of a fixed point argument. Let θ_0^n be an arbitrary nonzero element of $\hat{\theta}_b^n + U^n$ and define, for $j \in \mathbb{N}$, the function $\theta_j^n \in \hat{\theta}_b^n + U^n$ as follows: given θ_{j-1}^n we first find $(\mathbf{S}_j^n, \boldsymbol{u}_j^n, p_j^n) \in \Sigma^n \times V^n \times M_0^n$ by solving

$$\begin{aligned}
\int_\Omega (\boldsymbol{\mathcal{D}}^n(\mathbf{S}_j^n, \theta_{j-1}^n) - \mathbf{D}(\boldsymbol{u}_j^n)) : \boldsymbol{\tau} &= 0 && \forall \boldsymbol{\tau} \in \Sigma^n,\\
\int_\Omega \left(\frac{1}{j}\mathbf{D}(\boldsymbol{u}_j^n) + \mathbf{S}_j^n\right) : \mathbf{D}(\boldsymbol{v}) + \mathcal{B}(\boldsymbol{u}_j^n, \boldsymbol{u}_j^n, \boldsymbol{v}) - \int_\Omega p_j^n \operatorname{div} \boldsymbol{v} &= \int_\Omega \theta_{j-1}^n \boldsymbol{v} \cdot \boldsymbol{e}_d && \forall \boldsymbol{v} \in V^n,\\
-\int_\Omega q \operatorname{div} \boldsymbol{u}_j^n &= 0 && \forall q \in M^n,
\end{aligned} \tag{5.84}$$

and then θ_j^n is defined as $\hat{\theta}_b^n + \tilde{\theta}_j^n$, where $\tilde{\theta}_j^n \in U^n$ is the solution of the nonlinear problem

$$\int_\Omega \kappa(\tilde{\theta}_j^n + \hat{\theta}_b^n)\nabla(\tilde{\theta}_j^n + \hat{\theta}_b^n) \cdot \nabla\eta + \mathcal{C}(\boldsymbol{u}_j^n, \tilde{\theta}_j^n + \hat{\theta}_b^n, \eta) = 0 \qquad \forall \eta \in U^n. \tag{5.85}$$

In order to show that the problem (5.84) is well-posed, let us define a mapping $F_j^n : \Sigma^n \times V_{\mathrm{div}}^n \to (\Sigma^n \times V_{\mathrm{div}}^n)^*$ by

$$\begin{aligned}
\langle F_j^n(\boldsymbol{\sigma}, \boldsymbol{v}); (\boldsymbol{\tau}, \boldsymbol{w})\rangle := \int_\Omega (\boldsymbol{\mathcal{D}}^n(\boldsymbol{\sigma}, \theta_{j-1}^n) : \boldsymbol{\tau} - \mathbf{D}(\boldsymbol{v}) : \boldsymbol{\tau} + \frac{1}{j}\mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{w})\\
+ \boldsymbol{\sigma} : \mathbf{D}(\boldsymbol{w}) + \mathcal{B}(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{w}) - \theta_j^n \boldsymbol{v} \cdot \boldsymbol{e}_d).
\end{aligned}$$

By using the coercivity of $\boldsymbol{\mathcal{D}}^n$ and the fact that $\mathcal{B}(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v}) = 0$, one obtains using the inequalities of Young, Korn and Poincaré that there exists a $\delta(j) > 0$ such that

$$\langle F_j^n(\boldsymbol{\sigma}, \boldsymbol{v}), (\boldsymbol{\sigma}, \boldsymbol{v})\rangle > 0 \quad \text{if} \quad \|(\boldsymbol{\sigma}, \boldsymbol{v})\| = \delta(j).$$

A corollary of Brouwer's fixed point theorem [GR86, Ch. 4, Cor. 1.1] guarantees the existence of functions $(\mathbf{S}_j^n, \boldsymbol{u}_j^n) \in \Sigma^n \times V_{\mathrm{div}}^n$ satisfying $F_j^n(\mathbf{S}_j^n, \boldsymbol{u}_j^n) = 0$ (which is equivalent to (5.84) with divergence-free test functions) and such that $\|(\mathbf{S}_j^n, \boldsymbol{u}_j^n)\| \leq \delta(j)$. The existence of $p_j^n \in M_0^n$ then follows from the inf-sup condition (2.22). A similar argument can be used to prove the well-posedness of the problem (5.85).

Now, the inf-sup condition (2.23) and the discrete form of the constitutive relation (5.82a) allow us to control, uniformly in j and n, the norm of the velocity in terms of the stress:

$$\gamma_2 \|\boldsymbol{u}_j^n\|_{H^1(\Omega)} \leq \|\mathbf{S}_j^n\|_{L^2(\Omega)}. \tag{5.86}$$

Therefore, testing (5.84) with $(\mathbf{S}_j^n, \boldsymbol{u}_k^n, p_j^n)$ yields the estimate

$$\|\boldsymbol{\mathcal{D}}^n(\mathbf{S}_j^n, \theta_{j-1}^n)\|_{L^2(\Omega)}^2 + \|\mathbf{S}_j^n\|_{L^2(\Omega)}^2 + \|\boldsymbol{u}_j^n\|_{H^1(\Omega)}^2 \leq c\|\theta_{j-1}^n\|_{L^2(\Omega)}^2, \tag{5.87}$$

where $c > 0$ is independent of j and n. The inf-sup condition (2.22) and the discrete momentum equation in turn imply an estimate for the pressure:

$$\|p_j^n\|_{L^2(\Omega)}^2 \leq c\|\theta_{j-1}^n\|_{L^2(\Omega)}^2. \tag{5.88}$$

Furthermore, testing (5.85) with $\theta_j^n - \hat{\theta}_b^n$ results in

$$\|\theta_j^n\|_{H^1(\Omega)}^2 \leq c\|\boldsymbol{u}_j^n\|_{H^1(\Omega)}^2. \tag{5.89}$$

Hence, up to a subsequence, we have as $j \to \infty$ that

$$\begin{aligned}
\boldsymbol{\mathcal{D}}^n(\mathbf{S}_j^n, \theta_{j-1}^n) &\rightharpoonup \overline{\mathbf{D}}^n && \text{weakly in } L^2_{\text{sym}}(\Omega)^{d\times d}, \\
\mathbf{S}_j^n &\to \mathbf{S}^n && \text{strongly in } L^2_{\text{sym}}(\Omega)^{d\times d}, \\
\boldsymbol{u}_j^n &\to \boldsymbol{u}^n && \text{strongly in } H^1(\Omega)^d, \\
p_j^n &\to p^n && \text{strongly in } L^2(\Omega), \\
\theta_j^n &\to \theta^n && \text{strongly in } H^1(\Omega),
\end{aligned} \tag{5.90}$$

where we used the fact that weak and strong convergence are equivalent in finite-dimensional spaces. Since $\boldsymbol{\mathcal{D}}^n$ is continuous and the convergences are strong, one can straightforwardly identify $\overline{\mathbf{D}}^n = \boldsymbol{\mathcal{D}}^n(\mathbf{S}^n, \theta^n)$ and pass to the limit to show that $(\mathbf{S}^n, \theta^n, \boldsymbol{u}^n, p^n)$ solve Formulation B_0^n. Now, testing Formulation B_0^n with $(\mathbf{S}^n, \theta^n - \hat{\theta}_b^n, \boldsymbol{u}^n, p^n)$ allows one to obtain the estimate (5.83). Note that the inf-sup conditions were essential to obtain estimates that are uniform in n. $\square$

Having shown that the discrete problems are well-posed, we now consider the question of convergence.

Theorem 5.2.3. *Suppose the same assumptions as in Lemma 5.2.2 hold and suppose that $\{(\theta^n, \boldsymbol{u}^n, p^n)\}_{n\in\mathbb{N}}$ (respectively $(\{\mathbf{S}^n, \theta^n, \boldsymbol{u}^n, p^n\}_{\mathbb{N}})$) is a sequence of solutions of Formulation A_0^n or $\tilde{\mathrm{A}}_0^n$ (resp. Formulation B_0^n or $\tilde{\mathrm{B}}_0^n$). Then there exists a solution*

$(\mathbf{S}, \theta, \boldsymbol{u}, p) \in L^2_{\mathrm{sym,tr}}(\Omega)^{d\times d} \times (\hat{\theta}_b + H^1_{\Gamma_D}(\Omega)) \times H^1_0(\Omega)^d \times L^2_0(\Omega)$ *of Formulation* A_0 *such that, up to a subsequence, as* $n \to \infty$*:*

$$
\begin{aligned}
\mathbf{S}^n &\rightharpoonup \mathbf{S} && \textit{weakly in } L^2_{\mathrm{sym}}(\Omega)^{d\times d},\\
\boldsymbol{u}^n &\rightharpoonup \boldsymbol{u} && \textit{weakly in } H^1(\Omega)^d,\\
p^n &\rightharpoonup p && \textit{weakly in } L^2(\Omega),\\
\theta^n &\rightharpoonup \theta && \textit{weakly in } H^1(\Omega),
\end{aligned} \tag{5.91}
$$

where in the case of Formulations A^n_0 *and* $\tilde{\mathrm{A}}^n_0$ *we denote* $\mathbf{S}^n := \boldsymbol{\mathcal{S}}^n(\mathbf{D}(\boldsymbol{u}^n), \theta^n)$*.*

Proof. We will once again focus on Formulation B^n_0, since the other cases are completely analogous. From the a priori estimate (5.83) and the fact that $\hat{\theta}^n_b \to \hat{\theta}_b$ in $H^1(\Omega)$, we immediately obtain the convergences (5.91) (for a not relabelled subsequence) for some $(\mathbf{S}, \theta, \boldsymbol{u}, p) \in L^2_{\mathrm{sym}}(\Omega)^{d\times d} \times (\hat{\theta}_b + H^1_{\Gamma_D}(\Omega)) \times H^1_0(\Omega)^d \times L^2_0(\Omega)$, and that

$$
\boldsymbol{\mathcal{D}}^n(\mathbf{S}^n, \theta^n) \rightharpoonup \overline{\mathbf{D}} \quad \text{weakly in } L^2_{\mathrm{sym}}(\Omega)^{d\times d}. \tag{5.92}
$$

All that is left to prove is that the limiting functions are a solution of Formulation A_0. Let $\boldsymbol{\tau} \in L^2_{\mathrm{sym}}(\Omega)^{d\times d}$ be arbitrary. Then (5.91) and (2.21) result in

$$
0 = \int_\Omega (\boldsymbol{\mathcal{D}}^n(\mathbf{S}^n, \theta^n) - \mathbf{D}(\boldsymbol{u}^n)) : \Pi^n_\Sigma \boldsymbol{\tau} \xrightarrow[n\to\infty]{} \int_\Omega (\overline{\mathbf{D}} - \mathbf{D}(\boldsymbol{u})) : \boldsymbol{\tau}, \tag{5.93}
$$

and therefore $\overline{\mathbf{D}} = \mathbf{D}(\boldsymbol{u})$ almost everywhere. Similarly, for an arbitrary $q \in L^2_0(\Omega)$ one obtains that

$$
0 = \int_\Omega \operatorname{div} \boldsymbol{u}^n \, \Pi^n_M q \xrightarrow[n\to\infty]{} \int_\Omega \operatorname{div} \boldsymbol{u} \, q, \tag{5.94}
$$

and so $\boldsymbol{u}$ is pointwise divergence-free. One can pass to the limit in (5.82b) and (5.82d) in a similar manner, but perhaps the convective terms are worth looking at in more detail. To that end, first note that the Sobolev embedding theorem ensures that (up to a subsequence) we have, for any $p \in [1, 2^*)$,

$$
\begin{aligned}
\boldsymbol{u}^n &\to \boldsymbol{u} && \text{strongly in } L^p(\Omega)^d,\\
\theta^n &\to \theta && \text{strongly in } L^p(\Omega),\\
\theta^n &\to \theta && \text{a.e. in } \Omega.
\end{aligned} \tag{5.95}
$$

The strong convergence of $\boldsymbol{u}^n$ suffices to prove that, for an arbitrary $\boldsymbol{v} \in H^1_0(\Omega)^d$:

$$
\mathcal{B}(\boldsymbol{u}^n, \boldsymbol{u}^n, \Pi^n_V \boldsymbol{v}) \xrightarrow[n\to\infty]{} \frac{1}{2}\int_\Omega \boldsymbol{u}\otimes\boldsymbol{v} : \nabla\boldsymbol{u} - \boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{v} = -\int_\Omega \boldsymbol{u}\otimes\boldsymbol{u} : \mathbf{D}\boldsymbol{v}, \tag{5.96}
$$

where the last equality is a consequence of the fact that div $\boldsymbol{u} = 0$. Now, from testing the discrete momentum equation with $\boldsymbol{u}^n$ and taking (5.95) into account we observe that

$$\limsup_{n\to\infty} \int_\Omega \mathbf{S}^n : \mathbf{D}(\boldsymbol{u}^n) = \lim_{n\to\infty} \int_\Omega \theta^n \boldsymbol{u}^n \cdot \boldsymbol{e}_d = \int_\Omega \theta \boldsymbol{u} \cdot \boldsymbol{e}_d = \int_\Omega \mathbf{S} : \mathbf{D}(\boldsymbol{u}), \tag{5.97}$$

and hence by Lemma 5.1.2 we conclude that $(\mathbf{D}(\boldsymbol{u}), \mathbf{S}, \theta) \in \mathcal{A}$. Finally, by taking traces on both sides of the constitutive relation we also obtain that $\operatorname{tr}\mathbf{S} = 0$ and so $\mathbf{S} \in L^2_{\text{sym,tr}}(\Omega)^{d\times d}$, which concludes the proof. $\square$

Just as with Theorem 5.1.5, in the proof of Theorem 5.2.3 it becomes clear that the only bottleneck that prevents one from considering constitutive laws with more general r-coercivity (e.g. a power-law with temperature dependent consistency), is the fact that Lemma 5.1.2 is tied to the particular constitutive relation defined in (5.5). Using Minty's trick it is possible to show that if an explicit constitutive relation is available, an analogous convergence result will hold.

Assumption 5.2.4. *Let $\boldsymbol{\mathcal{S}} : \Omega\times\mathbb{R}^{d\times d}_{\text{sym}}\times\mathbb{R} \to \mathbb{R}^{d\times d}_{\text{sym}}$ be a continuous function satisfying, for some $r > \frac{2d}{d+2}$:*

- *(Monotonicity). For every $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{R}^{d\times d}_{\text{sym}}$:*

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{\tau}_1, s) - \boldsymbol{\mathcal{S}}(\boldsymbol{\tau}_2, s)) : (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2) \geq 0 \text{ for fixed } s \in \mathbb{R};$$

- *(Coercivity). There is a non-negative function $m \in L^1(\Omega)$ and a constant $c > 0$ such that*

$$\boldsymbol{\mathcal{S}}(\boldsymbol{\tau}, s) : \boldsymbol{\tau} \geq -m + c(|\boldsymbol{\mathcal{S}}(\boldsymbol{\tau}, s)|^{r'} + |\boldsymbol{\tau}|^r) \quad \text{for all } \boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{\text{sym}}, s \in \mathbb{R};$$

- *(Growth). There is a function $n \in L^{r'}(\Omega)$ and a constant $c > 0$ such that*

$$|\boldsymbol{\mathcal{S}}(\boldsymbol{\tau}, s)| \leq c(|\boldsymbol{\tau}|^{r'-1} + n);$$

- *(Compatibility). For a fixed $s \in \mathbb{R}$ we have that $\operatorname{tr}(\boldsymbol{\mathcal{S}}(\boldsymbol{\tau}, s)) = 0$ if and only if $\operatorname{tr}(\boldsymbol{\tau}) = 0$, for any $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{\text{sym}}$.*

When $r < \frac{3d}{d+2}$ the velocity $\boldsymbol{u}$ is not an admissible test function anymore and so obtaining an identity such as (5.97) is not straightforward. This difficulty can be overcome by testing instead with a discrete Lipschitz truncation of the error $\boldsymbol{e}^n := \boldsymbol{u} - \boldsymbol{u}^n$. The discrete Lipschitz truncation was introduced in [DKS13], and the idea is that it turns $\boldsymbol{e}^n$ into a Lipschitz function belonging to V^n in such a way that the size of the set where the truncation does not equal the original function can be controlled. We note that the construction of this discrete Lipschitz truncation requires a refined version of Assumption 2.5.3.

Assumption 5.2.5 (Fortin Projector Π_V^n)**.** *For each $n \in \mathbb{N}$ there is a linear projector $\Pi_V^n \colon W_0^{1,1}(\Omega)^d \to V^n$ such that it preserves the divergence in the same sense as in Assumption 2.5.3, but the stability condition is replaced by:*

- *(Local $W^{1,1}$-stability). For every $s \in (1,\infty)$ there is a constant $c > 0$, independent of n, such that*
$$\frac{1}{|K|}\int_K |\nabla \Pi_V^n \boldsymbol{v}| \leq c\frac{1}{|\Omega_K^n|}\int_{\Omega_K^n} |\nabla \boldsymbol{v}| \qquad \forall\, \boldsymbol{v} \in W_0^{1,s}(\Omega)^d, K \in \mathcal{T}_n,$$
where Ω_K^n denotes the patch of elements in $\mathcal{T}_n$ whose intersection with K is nonempty.

It can be shown that the local $W^{1,1}$-stability from Assumption 5.2.5 implies the global $W^{1,s}$-stability of Assumption 2.5.3 [BBDR12, DKS13]. Some examples of finite elements satisfying Assumption 5.2.5 include the conforming Crouzeix–Raviart element, the MINI element, the Bernardi–Raugel element, the $\mathbb{P}_2$–$\mathbb{P}_0$ and the Taylor–Hood pair $\mathbb{P}_k$–$\mathbb{P}_{k-1}$ for $k \geq d$ [BBDR12]; the lowest order Taylor–Hood pair in 3D also satisfies the assumption if the mesh has a certain macroelement structure [GNS15]. As for exactly divergence-free elements, this assumption can also be verified for low order Guzmán–Neilan elements and the Scott–Vogelius pair, under certain restrictions on the mesh [DKS13, Tsc18].

Corollary 5.2.6. *Let $r > \frac{2d}{d+2}$ and let $\boldsymbol{\mathcal{S}} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ be a function satisfying Assumption 5.2.4 and suppose that $\{U^n, V^n, M^n\}_{n\in\mathbb{N}}$ is a family of finite element subspaces satisfying Assumptions 2.5.1, 2.5.4, 2.5.5, 5.2.1, and 5.2.5. Then, for any $n \in \mathbb{N}$, the finite element formulation obtained by replacing $\boldsymbol{\mathcal{S}}^n$ by $\boldsymbol{\mathcal{S}}$ in Formulation $\tilde{\mathrm{A}}_0^n$ admits a solution $(\theta^n, \boldsymbol{u}^n, p^n) \in (\hat{\theta}_b^n + U^n) \times V^n \times M_0^n$ and we have, up to subsequences, that*

$$\begin{aligned}
\boldsymbol{u}^n &\rightharpoonup \boldsymbol{u} && \text{weakly in } W^{1,r}(\Omega)^d,\\
p^n &\rightharpoonup p && \text{weakly in } L^{\tilde{r}}(\Omega),\\
\theta^n &\rightharpoonup \theta && \text{weakly in } H^1(\Omega),\\
\boldsymbol{\mathcal{S}}(\mathbf{D}(\boldsymbol{u}^n),\theta^n) &\rightharpoonup \mathbf{S} && \text{weakly in } L^{r'}_{\mathrm{sym}}(\Omega)^{d\times d},
\end{aligned}$$

where $(\mathbf{S},\theta,\boldsymbol{u},p) \in L^{r'}_{\mathrm{sym,tr}}(\Omega)^{d\times d} \times (\hat{\theta}_b + H^1_{\Gamma_D}(\Omega)) \times W_0^{1,r}(\Omega)^d \times L_0^{\tilde{r}}(\Omega)$ is a solution of Formulation A_0.

Proof. The proof is entirely analogous to the proofs of Lemma 5.2.2 and Theorem 5.2.3, with a few small differences. Firstly, the a priori estimate (5.83) changes to

$$\|\boldsymbol{u}^n\|_{W^{1,r}(\Omega)^d} + \|\theta^n\|_{H^1(\Omega)} + \|p^n\|_{L^{\tilde{r}}(\Omega)} + \|\mathbf{S}^n\|_{L^{r'}(\Omega)} \leq c, \tag{5.98}$$

which implies the desired weak convergences. On the other hand, since $r > \frac{2d}{d+2}$, for a small enough $\varepsilon > 0$ we have that $r > \frac{(2+\varepsilon)d}{d+(2+\varepsilon)}$, which implies that $\boldsymbol{u}^n \to \boldsymbol{u}$ strongly in $L^{2+\varepsilon}(\Omega)^d$ as $n \to \infty$. Furthermore, from the consistency condition in Assumption 5.2.1 we see that

$$\|\pi^n \boldsymbol{u}^n - \boldsymbol{u}\|_{L^{2+\varepsilon}(K)} \leq \|\boldsymbol{u}^n - \boldsymbol{u}\|_{L^{2+\varepsilon}(K)} + c h_K^{1+d(\frac{1}{2+\varepsilon}-\frac{1}{r})} \|\boldsymbol{u}^n\|_{W^{1,r}(K)},$$

where we have used a standard local inverse inequality; the exponent of h_K is positive by the choice of ε, which implies that $\pi^n \boldsymbol{u}^n \to \boldsymbol{u}$ strongly in $L^{2+\varepsilon}(\Omega)^d$ as $n \to \infty$. This is enough to pass to the limit in the convective term:

$$\tilde{\mathcal{B}}_n(\boldsymbol{u}^n, \boldsymbol{u}^n, \Pi^n \boldsymbol{v}) \xrightarrow[n\to\infty]{} -\int_\Omega \boldsymbol{u} \otimes \boldsymbol{u} : \mathbf{D}(\boldsymbol{v}), \tag{5.99}$$

for any $\boldsymbol{v} \in W_0^{1,(\frac{2+\varepsilon}{2})'}(\Omega)^d$. As for the identification of the constitutive relation, by testing the discrete momentum equation with the discrete Lipschitz truncation of the error $\boldsymbol{e}^n := \boldsymbol{u} - \boldsymbol{u}^n$ it is possible to prove that (see [Tsc18] for a similar argument)

$$\limsup_{n\to\infty} \int_\Omega \boldsymbol{\mathcal{S}}(\mathbf{D}(\boldsymbol{u}^n), \theta^n) : \mathbf{D}(\boldsymbol{u}^n) \leq \int_\Omega \mathbf{S} : \mathbf{D}(\boldsymbol{u}). \tag{5.100}$$

Furthermore, from the growth condition of $\boldsymbol{\mathcal{S}}$ and the dominated convergence theorem (note that, up to a subsequence, we have that $\theta^n \to \theta$ almost everywhere, c.f. (5.95)) we see, that for any $\boldsymbol{\tau} \in L^r_{\mathrm{sym}}(\Omega)^{d\times d}$,

$$\boldsymbol{\mathcal{S}}(\boldsymbol{\tau}, \theta^n) \to \boldsymbol{\mathcal{S}}(\boldsymbol{\tau}, \theta) \quad \text{strongly in } L^{r'}(\Omega)^{d\times d}, \tag{5.101}$$

as $n \to \infty$. Combining the monotonicity of $\boldsymbol{\mathcal{S}}$ with (5.100) and (5.101) yields for an arbitrary $\boldsymbol{\tau} \in L^r_{\mathrm{sym}}(\Omega)^{d\times d}$:

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \int_\Omega (\boldsymbol{\mathcal{S}}(\mathbf{D}(\boldsymbol{u}^n), \theta^n) - \boldsymbol{\mathcal{S}}(\boldsymbol{\tau}, \theta^n)) : (\mathbf{D}(\boldsymbol{u}^n) - \boldsymbol{\tau}) \\ &\leq \int_\Omega (\mathbf{S} - \boldsymbol{\mathcal{S}}(\boldsymbol{\tau}, \theta)) : (\mathbf{D}(\boldsymbol{u}^n) - \boldsymbol{\tau}). \end{aligned}$$

Choosing $\boldsymbol{\tau} = \mathbf{D}(\boldsymbol{u}) \pm \varepsilon \boldsymbol{\sigma}$ with an arbitrary $\boldsymbol{\sigma} \in C_0^\infty(\Omega)^{d\times d}$ and letting $\varepsilon \to 0$ concludes the proof. $\square$

Remark 5.2.7. *The use of the discrete Lipschitz truncation is only necessary when the velocity $\boldsymbol{u}$ is not an admissible test function in the momentum equation, which occurs when $r < \frac{3d}{d+2}$. If $r \geq \frac{3d}{d+2}$ then one can substitute Assumption 5.2.5 with Assumption 2.5.3. It is also important to note that if the trilinear form $\mathcal{B}$ is used instead, the stronger assumption $r > \frac{2d}{d+1}$ is required (see (5.78)).*

Remark 5.2.8. *If the constitutive relation can be written in the form $\mathbf{D}(\boldsymbol{u}) = \boldsymbol{\mathcal{D}}(\mathbf{S}, \theta)$, where $\boldsymbol{\mathcal{D}}$ satisfies analogous conditions to the ones stated in Assumption 5.2.4, then the corresponding 4-field formulation will also satisfy an analogous convergence result. An example of a constitutive relation captured by these assumptions is the Ostwald–de Waele power-law model with $r > \frac{2d}{d+2}$:*

$$
\begin{aligned}
\boldsymbol{\mathcal{S}}(\mathbf{D}, \theta) &:= K(\theta)|\mathbf{D}|^{r-2}\mathbf{D}, \\
\boldsymbol{\mathcal{D}}(\mathbf{S}, \theta) &:= \frac{1}{K(\theta)}\left|\frac{\mathbf{S}}{K(\theta)}\right|^{r'-2}\mathbf{S},
\end{aligned}
$$

where $K : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $c_1 \leq K(s) \leq c_2$ for any $s \in \mathbb{R}$, where c_1, c_2 are two positive constants. If $r \leq 2$ the Herschel–Bulkley relation (5.72) is included as well.

If the rheological parameters are not temperature-dependent, the convergence result can cover more general constitutive relations defined by maximal monotone r-graphs. For this problem let us define Formulation C_0 in exactly the same way as Formulation A_0, but where now the graph $\mathcal{A}$ in (5.81d) is any maximal monotone r-graph satisfying (A1)–(A6). Formulation $\tilde{\mathrm{C}}_0^n$ is then defined in the same way as Formulation $\tilde{\mathrm{B}}_0^n$, but with $\boldsymbol{\mathcal{D}}^n(\mathbf{S}^n, \theta^n)$ replaced by the generalised Yosida approximation (2.19). However, it is worth pointing out that in the numerical computations one can simply work with the implicit function directly by writing

$$
\int_\Omega \mathbf{G}(\cdot, \mathbf{S}^n, \mathbf{D}(\boldsymbol{u}^n)) : \boldsymbol{\tau} = 0 \quad \forall\, \boldsymbol{\tau} \in \Sigma^n, \tag{5.102}
$$

instead of (5.82a).

Corollary 5.2.9. *Let $r > \frac{2d}{d+2}$ and let $\mathcal{A}(\cdot) \subset \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$ be a graph satisfying (A1)–(A6). Suppose that $\{\Sigma^n, U^n, V^n, M^n\}_{n\in\mathbb{N}}$ is a family of finite element subspaces satisfying Assumptions 2.5.1, 2.5.2, 2.5.4, 2.5.5, 5.2.1, and 5.2.5. Then, for any $n \in \mathbb{N}$, Formulation $\tilde{\mathrm{C}}_0^n$ admits a solution $(\mathbf{S}^n, \theta^n, \boldsymbol{u}^n, p^n) \in \Sigma^n \times (\hat{\theta}_b^n + U^n) \times V^n \times M^n$,* *and we have, up to subsequences, that*

$$\begin{aligned}
\boldsymbol{u}^n &\rightharpoonup \boldsymbol{u} && \textit{weakly in } W^{1,r}(\Omega)^d,\\
p^n &\rightharpoonup p && \textit{weakly in } L^{\tilde{r}}(\Omega),\\
\theta^n &\rightharpoonup \theta && \textit{weakly in } H^1(\Omega),\\
\mathbf{S}^n &\rightharpoonup \mathbf{S} && \textit{weakly in } L^{r'}_{\mathrm{sym}}(\Omega)^{d\times d},
\end{aligned}$$

where $(\mathbf{S},\theta,\boldsymbol{u},p)\in L^{r'}_{\mathrm{sym,tr}}(\Omega)^{d\times d}\times(\hat{\theta}_b+H^1_{\Gamma_D}(\Omega))\times W^{1,r}_0(\Omega)^d\times L^{\tilde{r}}_0(\Omega)$ *is a solution of Formulation* C_0*.*

Remark 5.2.10. *When restricted to the isothermal case, the convergence result from Corollary 5.2.9 improves the one presented in [DKS13] in two respects: the graph is not required to be strictly monotone here, which allows models with a yield stress, for instance, and the result holds for the whole admissible range* $r>\frac{2d}{d+2}$ *even without the use of pointwise divergence-free elements, thanks to the modified convective term* $\tilde{\mathcal{B}}_n$*. In addition, the argument used here in the identification of the constitutive relation avoids the use of Young measures, simplifying the proof.*

5.2.2 Augmented Lagrangian Preconditioner

Henceforth we employ the Scott–Vogelius pair for the velocity and pressure, and discontinuous and continuous elements for the stress and temperature, respectively, with $k\geq d$:

$$\begin{aligned}
\Sigma^h &= \{\boldsymbol{\sigma}\in L^{\infty}_{\mathrm{sym,tr}}(\Omega)^{d\times d} \,:\, \boldsymbol{\sigma}|_K\in\mathbb{P}_{k-1}(K)^{d\times d} \text{ for all } K\in\mathcal{T}_n\},\\
U^h &= \{\eta\in W^{1,\infty}_{\Gamma_D}(\Omega) \,:\, \eta|_K\in\mathbb{P}_{k-1}(K) \text{ for all } K\in\mathcal{T}_n\},\\
V^h &= \{\boldsymbol{w}\in W^{1,\infty}_0(\Omega)^d \,:\, \boldsymbol{w}|_K\in\mathbb{P}_{k}(K)^d \text{ for all } K\in\mathcal{T}_n\},\\
M^h &= \{q\in L^{\infty}_0(\Omega) \,:\, q|_K\in\mathbb{P}_{k-1}(K) \text{ for all } K\in\mathcal{T}_n\}.
\end{aligned}\tag{5.103}$$

In order to ensure the inf-sup stability of the velocity-pressure pair, each level $\mathcal{T}_n$ of the mesh hierarchy is barycentrically refined, with the hierarchy itself constructed by uniform refinement, to prevent the appearance of degenerate elements (see Figure 4.2). As seen in Chapter 4, a drawback of this approach is that the resulting mesh hierarchy is non-nested, which introduces some difficulties when dealing with the transfer operators in the multigrid algorithm.

This choice of finite element space for the stress satisfies the inf-sup condition (2.23). In addition, recall that this also allows one to work with traceless stresses and hence fewer degrees of freedom will be required (see Remark 2.5.8). This exact enforcement of the divergence constraint was one of the motivations behind our choice of elements; it is known that a failure to enforce the divergence-free constraint appropriately can lead to unphysical behaviour in the solution of buoyancy-driven flow [JLM+17].

At this point the viscous dissipation and the adiabatic heating terms can be incorporated into the formulation. For instance, when working with the setting described by Corollary 5.2.6, in the finite element formulation we seek $(\theta^n, \boldsymbol{u}^n, p^n) \in (\hat{\theta}_b + U^n) \times V^n \times M_0^n$ such that

$$\begin{aligned}\int_\Omega \boldsymbol{\mathcal{S}}(\mathsf{D}(\boldsymbol{u}^n), \theta^n) : \mathsf{D}(\boldsymbol{v}) - \int_\Omega (\boldsymbol{u}^n \otimes \boldsymbol{u}^n) : \mathsf{D}(\boldsymbol{v}) - \int_\Omega p^n \operatorname{div} \boldsymbol{v} &= \int_\Omega \theta^n \boldsymbol{v} \cdot \boldsymbol{e}_d \ \forall\, \boldsymbol{v} \in V^n, \\ -\int_\Omega q \operatorname{div} \boldsymbol{u}^n = 0 & \qquad\qquad \forall\, q \in M^n,\end{aligned} \tag{5.104a}$$

$$\int_\Omega (\kappa(\theta^n)\nabla\theta^n - \boldsymbol{u}^n\theta^n) \cdot \nabla\eta + \int_\Omega \theta^n \boldsymbol{u}^n \cdot \boldsymbol{e}_d \eta = \int_\Omega \boldsymbol{\mathcal{S}}(\mathsf{D}(\boldsymbol{u}^n), \theta^n) : \mathsf{D}(\boldsymbol{u}^n)\eta \ \forall\, \eta \in U^n,$$

with analogous modifications for the other formulations. Note that the form of the convective term could be simplified since the elements are exactly divergence-free. The nonlinear finite element formulations are linearised using Newton's method; for instance, if the current guess for the solution of (5.104) is $(\tilde{\theta}, \tilde{\boldsymbol{u}}, \tilde{p})$, then the method is defined by the correction step $(\tilde{\theta}, \tilde{\boldsymbol{u}}, \tilde{p}) \mapsto (\tilde{\theta}, \tilde{\boldsymbol{u}}, \tilde{p}) + (\theta, \boldsymbol{u}, p)$ where $(\theta, \boldsymbol{u}, p)$ is the solution of a linear system whose matrix has the block structure

$$\begin{bmatrix} A_1 & C & 0 \\ E & A_2 & \tilde{B}^\top \\ 0 & \tilde{B} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \boldsymbol{u} \\ p \end{bmatrix}. \tag{5.105}$$

The blocks in (5.105) are defined through the linear operators:

$$\begin{aligned}
\langle A_1\theta, \eta\rangle &:= \int_\Omega \theta\kappa'(\tilde{\theta})\nabla\tilde{\theta} \cdot \nabla\eta + \int_\Omega \kappa(\tilde{\theta})\nabla\theta \cdot \nabla\eta - \int_\Omega \tilde{\boldsymbol{u}}\theta \cdot \nabla\eta \\
&\quad + \int_\Omega \tilde{\boldsymbol{u}}\theta \cdot \boldsymbol{e}_d\eta - \int_\Omega \boldsymbol{\mathcal{S}}_\theta(\mathsf{D}(\tilde{\boldsymbol{u}}), \tilde{\theta}) : \mathsf{D}(\tilde{\boldsymbol{u}})\theta\eta & \forall\, \theta, \eta \in U^n, \\
\langle C\boldsymbol{u}, \eta\rangle &:= \int_\Omega \tilde{\theta}\boldsymbol{u} \cdot (\boldsymbol{e}_d\eta - \nabla\eta) - \int_\Omega \boldsymbol{\mathcal{S}}_{\mathsf{D}}(\mathsf{D}(\tilde{\boldsymbol{u}}), \tilde{\theta})\mathsf{D}(\boldsymbol{u}) : \mathsf{D}(\boldsymbol{v})\eta \\
&\quad - \int_\Omega \boldsymbol{\mathcal{S}}(\mathsf{D}(\tilde{\boldsymbol{u}}),\tilde{\theta}) : \mathsf{D}(\boldsymbol{u})\eta & \forall\, \boldsymbol{u} \in V^n, \eta \in U^n, \\
\langle E\theta, \boldsymbol{v}\rangle &:= \int_\Omega \boldsymbol{\mathcal{S}}_\theta(\mathsf{D}(\tilde{\boldsymbol{u}}), \tilde{\theta})\theta : \mathsf{D}(\boldsymbol{v}) - \int_\Omega \theta\boldsymbol{v} \cdot \boldsymbol{e}_d & \forall\, \theta \in U^n, \boldsymbol{v} \in V^n, \\
\langle A_2\boldsymbol{u}, \boldsymbol{v}\rangle &:= \int_\Omega \Big(\boldsymbol{\mathcal{S}}_{\mathsf{D}}(\mathsf{D}(\tilde{\boldsymbol{u}}), \tilde{\theta})\ \mathsf{D}(\boldsymbol{u}) - \tilde{\boldsymbol{u}} \otimes \boldsymbol{u} - \boldsymbol{u} \otimes \tilde{\boldsymbol{u}}\big) : \mathsf{D}(\boldsymbol{v}) & \forall\, \boldsymbol{u}, \boldsymbol{v} \in V^n, \\
\langle \tilde{B}\boldsymbol{v}, q\rangle &:= -\int_\Omega q \operatorname{div} \boldsymbol{v} & \forall\, \boldsymbol{v} \in V^n, q \in M^n.
\end{aligned}$$

We use the notation $\boldsymbol{\mathcal{S}}_{\mathsf{D}}, \boldsymbol{\mathcal{S}}_{\theta}$ to denote the partial derivatives of $\boldsymbol{\mathcal{S}}$; for instance, for the Navier–Stokes model one would have $\boldsymbol{\mathcal{S}}_{\mathsf{D}}(\mathsf{D}(\tilde{\boldsymbol{u}}), \tilde{\theta}) = 2\hat{\mu}(\tilde{\theta})\boldsymbol{I}$ and $\boldsymbol{\mathcal{S}}_{\theta}(\mathsf{D}(\tilde{\boldsymbol{u}}), \tilde{\theta}) = 2\hat{\mu}'(\tilde{\theta})\mathsf{D}(\tilde{\boldsymbol{u}})$, where $\boldsymbol{I}$ is the fourth-order identity tensor.

Keeping (5.105) as an illustrative example, we see that after augmentation the top block can be written in the form

$$A + \gamma B^\top M_p^{-1} B = \begin{bmatrix} A_1 & C \\ E & A_2 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ \tilde{B}^\top \end{bmatrix} M_p^{-1} \begin{bmatrix} 0 & \tilde{B} \end{bmatrix}, \tag{5.107}$$

where A is invertible and $\gamma B^\top M_p^{-1} B$ is symmetric and semi-definite. Let us define $Z^n := U^n \times V^n$ whenever the 3-field formulation is employed and $Z^n := \Sigma^n \times U^n \times V^n$ otherwise. Following the approach presented in Chapter 4, we consider the subspace decomposition based on macrostar patches:

$$Z^n = \sum_i \{\boldsymbol{z} \in Z^n \,:\, \mathrm{supp}(\boldsymbol{z}) \subset \mathrm{macrostar}(q_i)\}, \tag{5.108}$$

which ensures that the kernel of the semi-definite term is captured:

$$\mathcal{N}^n = \sum_i Z_i^n \cap \mathcal{N}^n. \tag{5.109}$$

Here $\mathcal{N}^n$ consists of the elements of the form $(\theta, \boldsymbol{v})^\top$ and $(\boldsymbol{\sigma}, \theta, \boldsymbol{v})^\top$ for the 3-field and 4-field formulations, respectively, where $\boldsymbol{v} \in V^n_{\mathrm{div}}$, and $\boldsymbol{\sigma} \in \Sigma^n$, $\theta \in U^n$ are arbitrary.

Regarding transfer operators, for the temperature we employ a standard prolongation based on interpolation, and for the velocity the prolongation operator defined by (4.33). For the formulations including the stress, we make use of the supermesh projection (4.36). An overview of the algorithm is shown in Figure 5.1.

5.3 Numerical experiments

Let us suppose that the parameters in the constitutive relation (5.5) can be written as

$$\frac{\mu(\theta)}{\mu_0} = \hat{\mu}(\theta), \quad \frac{\kappa(\theta)}{\kappa_0} = \hat{\kappa}(\theta), \quad \frac{\tau(\theta)}{\tau_0} = \hat{\tau}(\theta), \quad \frac{\sigma(\theta)}{\sigma_0} = \hat{\sigma}(\theta), \tag{5.110}$$

where $\mu_0, \kappa_0 > 0$ are reference values for the viscosity and heat conductivity, $\tau, \sigma \geq 0$ are reference values for the activation parameters, and $\hat{\mu}, \hat{\kappa}, \hat{\tau}, \hat{\sigma}$ are then non-dimensional functions. In practice the system can be non-dimensionalised in distinct ways to give more importance to different physical regimes. For example, suppose that

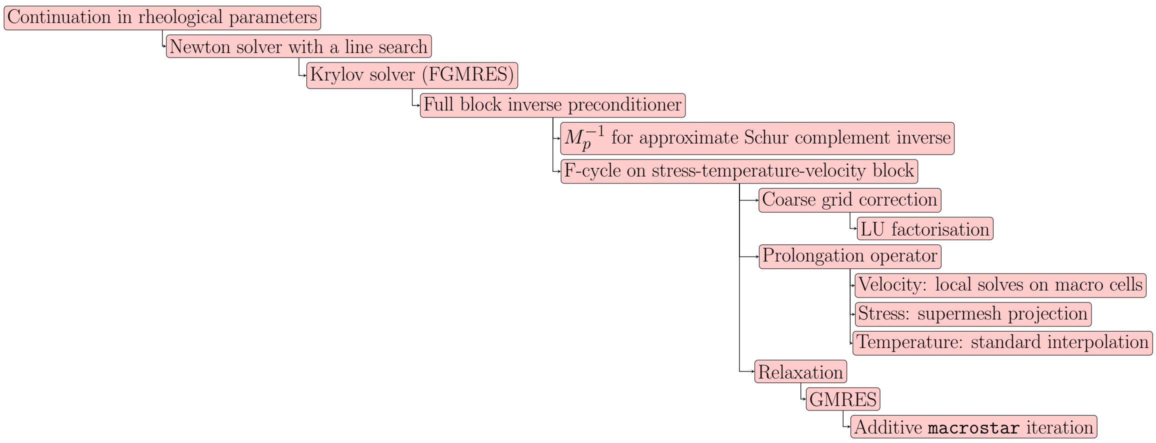


Figure 5.1: Overview of the algorithm for the steady buoyancy-driven flow.

the time scale is chosen based on the diffusion of heat, and that the non-dimensional variables are introduced in the following way:

$$\tilde{t} := \frac{\tilde{\alpha}}{L^2}t, \quad \tilde{x} := \frac{x}{L}, \quad \tilde{\boldsymbol{u}} := \frac{L}{\tilde{\alpha}}\boldsymbol{u}, \quad \tilde{p} := \frac{L^2}{\rho_* \tilde{\alpha}^2}p, \quad \tilde{\theta} := \frac{\theta - \theta_C}{\theta_H - \theta_C}, \quad \tilde{\mathbf{S}} := \frac{L^2}{\mu_0 \tilde{\alpha}}\mathbf{S}, \tag{5.111}$$

where L is a characteristic length scale, θ_H is a reference temperature (e.g. the temperature of the hot plate in a Bénard problem), and $\tilde{\alpha} = \frac{\kappa_0}{\rho_* c_p}$ is the thermal diffusion rate. Then, the non-dimensional form of the system reads (dropping the tildes):

$$-\mathrm{Pr}\,\mathrm{div}\,\mathbf{S} + \mathrm{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p = \mathrm{Ra}\,\mathrm{Pr}\,\theta\boldsymbol{e}_d \qquad \text{in } \Omega, \tag{5.112a}$$

$$\mathrm{div}\,\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{5.112b}$$

$$-\mathrm{div}(\hat{\kappa}(\theta)\nabla\theta) + \mathrm{div}(\boldsymbol{u}\theta) + \mathrm{Di}(\theta + \Theta)\boldsymbol{u}\cdot\boldsymbol{e}_d = \frac{\mathrm{Di}}{\mathrm{Ra}}\mathbf{S}:\mathbf{D}(\boldsymbol{u}) \qquad \text{in } \Omega, \tag{5.112c}$$

where the Rayleigh, Prandtl, Dissipation and Theta numbers are defined respectively as

$$\mathrm{Ra} = \frac{\beta g(\theta_H - \theta_C)L^3}{\nu\tilde{\alpha}}, \qquad \mathrm{Pr} = \frac{\nu_0}{\tilde{\alpha}}, \qquad \mathrm{Di} = \frac{\beta g L}{c_p}, \qquad \Theta = \frac{\theta_C}{\theta_H - \theta_C}, \tag{5.113}$$

where $\nu_0 := \frac{\mu_0}{\rho_*}$ is the reference kinematic viscosity (more non-dimensional numbers could arise with a non-Newtonian constitutive relation). Alternatively, if one assumes that the gravitational potential energy is completely transformed into kinetic energy [HMW75, Ost58], the characteristic velocity is chosen as $U = (gL\beta(\theta_H - \theta_C))^{1/2}$ and the resulting non-dimensional system becomes

$$-\frac{1}{\sqrt{\mathrm{Gr}}}\,\mathrm{div}\,\mathbf{S} + \mathrm{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p = \theta\boldsymbol{e}_d \qquad \text{in } \Omega, \tag{5.114a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{5.114b}$$

$$-\frac{1}{\mathrm{Pr}\sqrt{\mathrm{Gr}}} \operatorname{div}(\hat{\kappa}(\theta)\nabla\theta) + \operatorname{div}(\boldsymbol{u}\theta) + \mathrm{Di}(\theta + \Theta)\boldsymbol{u} \cdot \boldsymbol{e}_d = \frac{\mathrm{Di}}{\sqrt{\mathrm{Gr}}} \mathbf{S} : \mathbf{D}(\boldsymbol{u}) \qquad \text{in } \Omega, \tag{5.114c}$$

where the Grashof number is defined as

$$\mathrm{Gr} = \frac{gL^3\beta(\theta_H - \theta_C)}{\nu_0^2}. \tag{5.115}$$

In the following section we will test the solver using the different forms (5.112) and (5.114) with a heated cavity problem. The computational examples were implemented in Firedrake [RHM+16], and PCPATCH [FKMW19] (a recently developed tool for subspace decomposition in multigrid in PETSc [BAA+17]) was employed for the macrostar patch solves in the multigrid algorithm. The augmented Lagrangian parameter was set to $\gamma = 10^4$, and unless specified otherwise, the Newton solver was deemed to have converged when the Euclidean norm of the residual fell below 1×10^{-8} and the corresponding tolerance for the linear solver in 2D was set to 1×10^{-10} (1×10^{-8} in 3D). In the implementation the uniqueness of the pressure was enforced by orthogonalizing against the nullspace of constants in the Krylov solver, instead of enforcing a zero mean condition.

5.3.1 Heated cavity

The problem is solved on the unit square/cube $\Omega = (0,1)^d$ with boundary data

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \qquad \nabla\theta \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \setminus (\Gamma_H \cup \Gamma_C), \qquad \theta = \begin{cases} 1, & \text{on } \Gamma_H, \\ 0, & \text{on } \Gamma_C, \end{cases}$$

where $\Gamma_H := \{x_1 = 0\}$ and $\Gamma_C := \{x_1 = 1\}$. For the problems with temperature-dependent viscosity and conductivity we choose the following functional dependences:

$$\hat{\mu}(\theta) := \mathrm{e}^{-\frac{\theta}{10}}, \tag{5.116a}$$

$$\hat{\kappa}(\theta) := \frac{1}{2} + \frac{\theta}{2} + \theta^2. \tag{5.116b}$$

The viscosity defined by (5.116a) decreases with temperature, as is the case with most liquids [FG02]; heat conductivities of the form (5.116b) are a good fit for most liquid metals and gases [EL99]. Let us denote the problem solved with $\hat{\mu}(\theta) \equiv 1 \equiv \hat{\kappa}(\theta)$ by (P1), the one using (5.116a) and $\hat{\kappa}(\theta) \equiv 1$ by (P2), and by (P3) the one using both forms in (5.116).

A simple continuation algorithm was used to reach the different values of the parameters; for instance, the solution corresponding to a Rayleigh number Ra was used as an initial guess in Newton's method for the problem with $\mathrm{Ra}+\mathrm{Ra}_{\mathrm{step}}$, where $\mathrm{Ra}_{\mathrm{step}}$ is some predetermined step. In some cases (most notably shear-thinning fluids) the use of advective stabilisation was essential; here we added to the formulation the advective stabilisation term (4.43), with the same parameters described in Section 4.2.2. The choice of stabilisation was preferred over the more common SUPG stabilisation because the latter introduces additional couplings between the velocity and the pressure in the momentum equation, and between the velocity, stress and temperature in the energy equation, which can spoil the convergence of the nonlinear solver (this was already observed in the isothermal case in [FMSW20a]). The disadvantage is that (4.43) introduces an additional kernel consisting of C^1 functions, that might not be captured by the relaxation. This means that unless $k \geq 3$ in 2D or $k \geq 5$ in 3D, a slight loss of robustness might be expected [FMSW20a].

Tables 5.1–5.3 show the average number of Krylov iterations for the problem with non-dimensional form (5.114) and increasingly large Grashof number, comparing with different values of the Dissipation number; Tables 5.4–5.6 show the same for the three-dimensional problem. It can be observed that the iteration counts remain under control, and the previously mentioned loss of robustness occurs when $k = 2$. Figure 5.2 shows the streamlines and temperature contours for the problem (P2); it can be observed that the presence of the viscous dissipation term has a stabilising effect on the flow. Table 5.7 shows the number of iterations for the problem using the temperature-dependent power-law relation

$$\mathsf{S} = \boldsymbol{\mathcal{S}}(\mathsf{D}(\boldsymbol{u}), \theta) := \mathrm{e}^{-\frac{\theta}{10}} |\mathsf{D}(\boldsymbol{u})|^{r-2} \mathsf{D}(\boldsymbol{u}), \tag{5.117}$$

using $r = 1.6$ and the streamlines are shown in Figure 5.4 alongside the ones of the Newtonian problem ($r = 2$). In this case the tolerances for the linear and nonlinear iterations were set to 1×10^{-10}, and 7 multigrid cycles with 7 relaxation sweeps per level were employed. Admittedly, this is a computationally expensive solver whose applicability in a practical setting depends on the range of parameters one wishes to simulate.

Di	k	# refs	# dofs	Gr			
				1×10^6	5×10^6	1×10^7	1.5×10^7
0	2	1	1.8×10^4	5	7.66	10	22
		2	7.2×10^4	4.25	7	8	8.5
	3	1	4.1×10^4	2	3.5	4	4.5
		2	1.6×10^5	1.66	2	2.5	3
0.6	2	1	1.8×10^4	4.75	8	13.3	18.7
		2	7.2×10^4	4	7	7.5	7
	3	1	4.1×10^4	2	3.5	4.5	5.5
		2	1.6×10^5	1.67	2	3.5	4
1.3	2	1	1.8×10^4	5.67	8	12.67	18.67
		2	7.2×10^4	4	6.5	6.5	7
	3	1	4.1×10^4	2	2.5	4	4
		2	1.6×10^5	1.67	2	2.5	2.5
2.0	2	1	1.8×10^4	5.67	9.33	12.67	18.67
		2	7.2×10^4	4	6.5	6.5	8
	3	1	4.1×10^4	2	2.5	3	3
		2	1.6×10^5	1.67	2	2	2

Table 5.1: Average number of Krylov iterations per Newton step as Gr increases for the 2D problem (P1) with Pr = 1, obtained using 2 multigrid cycles with 4 relaxation sweeps.

Di	k	# refs	# dofs	Gr 1×10^6	5×10^6	1×10^7	1.25×10^7
0	2	1	1.8×10^4	5.25	8.33	18	23.25
		2	7.2×10^4	4.25	7.5	9	9.5
	3	1	4.1×10^4	2	3.5	4.5	5
		2	1.6×10^5	1.67	2	2.5	2.5
0.6	2	1	1.8×10^4	4.75	8.67	15	15.5
		2	7.2×10^4	4.33	7	7.5	7.5
	3	1	4.1×10^4	2.33	3.5	5.5	5.5
		2	1.6×10^5	1.67	2	3.5	4.5
1.3	2	1	1.8×10^4	4.75	9.33	15.67	20.67
		2	7.2×10^4	4	7	6.5	6.5
	3	1	4.1×10^4	2	3.5	4	4.5
		2	1.6×10^5	1.67	2	3	3.5
2.0	2	1	1.8×10^4	5.67	10.67	16.33	19
		2	7.2×10^4	4	7	6.5	7.5
	3	1	4.1×10^4	2	2.5	3	3
		2	1.6×10^5	1.67	2	2.5	2.5

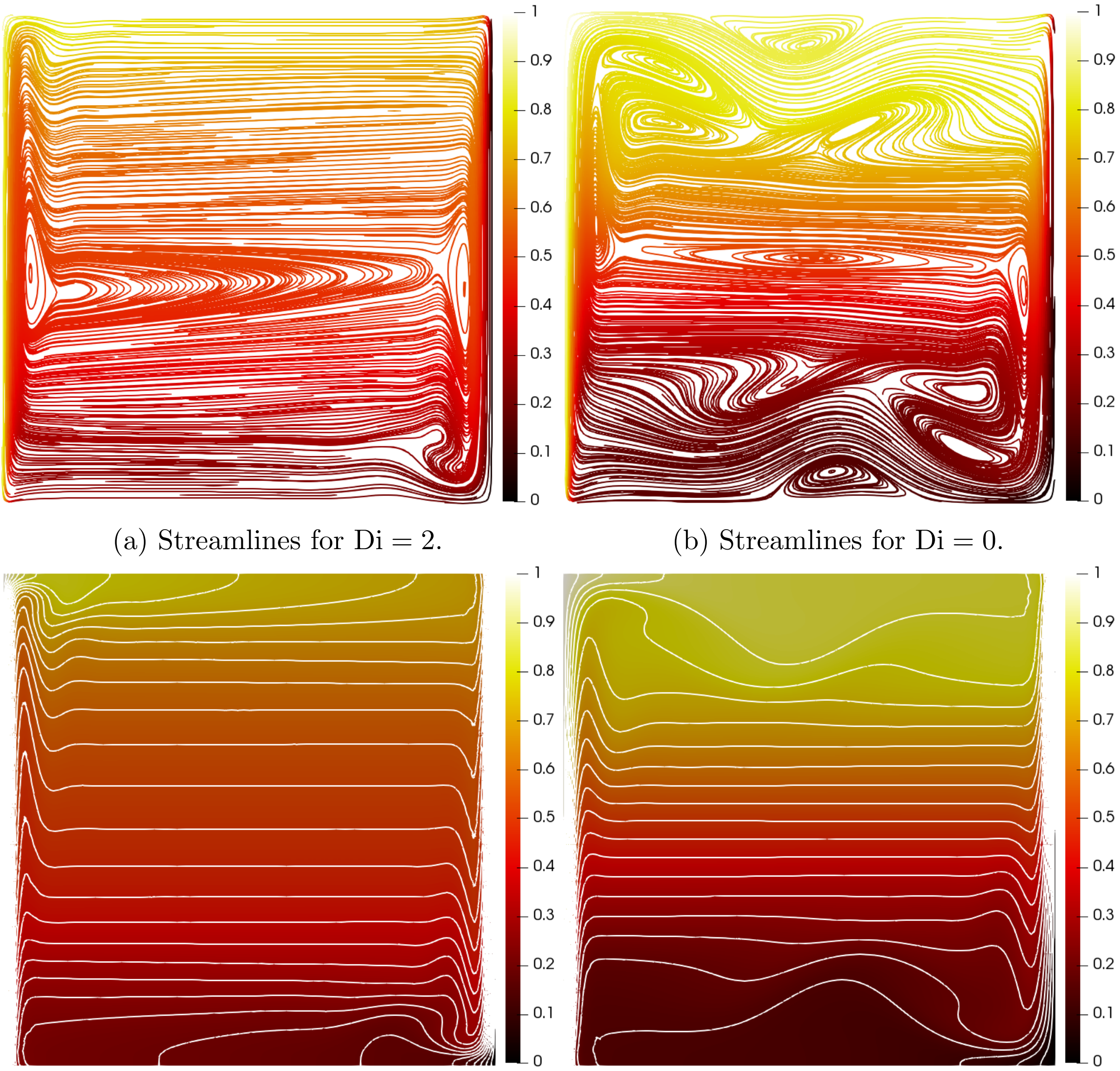
Table 5.2: Average number of Krylov iterations per Newton step as Gr increases for the 2D problem (P2) with Pr = 1, obtained using 2 multigrid cycles with 4 relaxation sweeps.

Di	k	# refs	# dofs	Gr			
				1×10^6	5×10^6	1×10^7	1.25×10^7
0	2	1	1.8×10^4	5.75	9	17.75	23
		2	7.2×10^4	4.25	6.33	10	11
	3	1	4.1×10^4	2	3	5	6
		2	1.6×10^5	1.67	2	1.5	1.5
0.6	2	1	1.8×10^4	5.5	9	17.33	24.4
		2	7.2×10^4	4.67	8	9	9.5
	3	1	4.1×10^4	2.33	3.5	5	6
		2	1.6×10^5	1.67	2.5	3.5	4
1.3	2	1	1.8×10^4	4.75	9.67	18	23.67
		2	7.2×10^4	4	8	9.5	9
	3	1	4.1×10^4	2.33	2.5	4	4
		2	1.6×10^5	1.67	2	2.5	2.5
2.0	2	1	1.8×10^4	5.66	10.33	18.33	24.33
		2	7.2×10^4	4.33	10	10	8.5
	3	1	4.1×10^4	2.33	2.5	2.5	3
		2	1.6×10^5	1.67	2	1.5	1.5

Table 5.3: Average number of Krylov iterations per Newton step as Gr increases for the 2D problem (P3) with Pr = 1, obtained using 2 multigrid cycles with 4 relaxation sweeps.

Di	# refs	# dofs	Gr			
			2.52×10^5	6.30×10^5	9.45×10^5	1.26×10^6
0	1	3.2×10^5	3.33	4	4.5	9
	2	2.6×10^6	6	4.5	3.5	3.5
0.6	1	3.2×10^5	3.33	4	4	10.5
	2	2.6×10^6	4.33	5	4.5	4.5
1.3	1	3.2×10^5	3.33	4	4	10.5
	2	2.6×10^6	6	4.5	4.5	4
2	1	3.2×10^5	3	4	4.5	12
	2	2.6×10^6	6	4.5	3.5	3.5

Table 5.4: Average number of Krylov iterations per Newton step as Gr increases for the 3D problem (P1) with Pr = 1 and $k = 3$, obtained using 2 multigrid cycles with 4 relaxation sweeps.



(a) Streamlines for Di = 2.

(b) Streamlines for Di = 0.

(c) Temperature contour for Di = 2.

(d) Temperature contour for Di = 0.

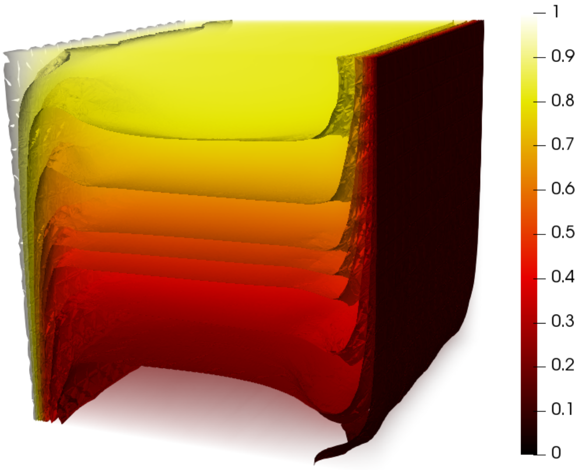
Figure 5.2: Streamlines and temperature contours for the heated cavity with temperature dependent viscosity and $\mathrm{Gr} = 1.25 \times 10^7$.

Di	# refs	# dofs	Gr			
			2.52×10^5	6.30×10^5	9.45×10^5	1.26×10^6
0	1	3.2×10^5	3.67	4	5	13.5
	2	2.6×10^6	5	5.5	5.5	5.5
0.6	1	3.2×10^5	3.33	4	4.5	14
	2	2.6×10^6	4.33	5.5	4.5	5
1.3	1	3.2×10^5	3.33	4	5	16.5
	2	2.6×10^6	6	4.5	4.5	4.5
2	1	3.2×10^5	3.67	4	5	13.5
	2	2.6×10^6	6	4.5	3.5	4

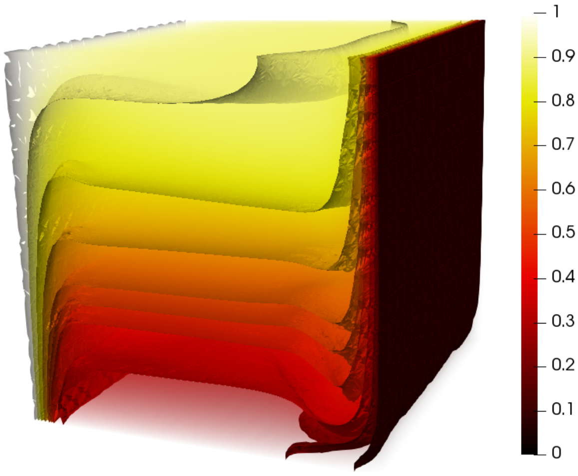
Table 5.5: Average number of Krylov iterations per Newton step as Gr increases for the 3D problem (P2) with Pr = 1 and $k = 3$, obtained using 2 multigrid cycles with 4 relaxation sweeps.

Di	# refs	# dofs	Gr			
			2.52×10^5	6.30×10^5	9.45×10^5	1.26×10^6
0	1	3.2×10^5	3.67	5	7.5	19
	2	2.6×10^6	5	6	6	9.5
0.6	1	3.2×10^5	3.33	4	4	10.5
	2	2.6×10^6	4.33	5.5	4.5	7
1.3	1	3.2×10^5	3.33	4	10	28.5
	2	2.6×10^6	6	4.5	4.5	4
2	1	3.2×10^5	3	4	11.5	41.5
	2	2.6×10^6	6	4.5	3.5	3.5

Table 5.6: Average number of Krylov iterations per Newton step as Gr increases for the 3D problem (P3) with Pr = 1 and $k = 3$, obtained using 2 multigrid cycles with 4 relaxation sweeps.



(a) Problem (P1).

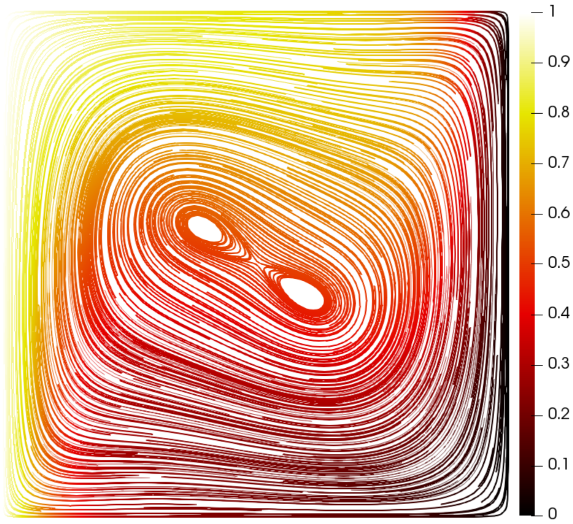


(b) Problem (P3).

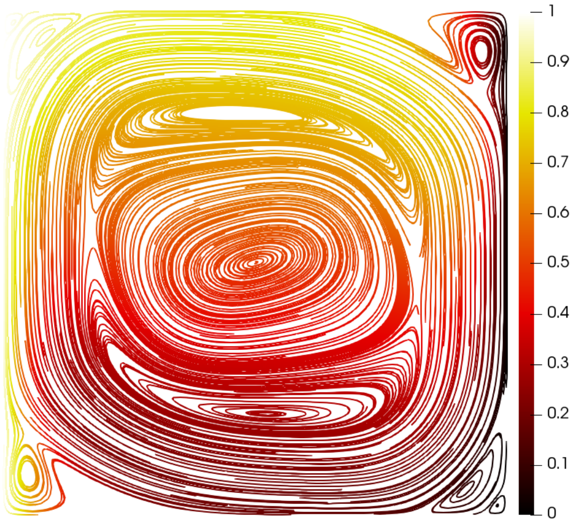
Figure 5.3: Temperature contours for the 3D heated cavity with $\mathrm{Gr} = 1.26 \times 10^6$.

k	# refs	# dofs	Ra 5000	Ra 10000	Ra 15000	Ra 20000
2	1	1.8×10^4	3.64	5.25	6.42	6.38
	2	7.2×10^4	3.78	5.78	7	9.75
	3	2.9×10^5	3.22	4.8	6.3	8.3
3	1	7.3×10^4	2.57	3.11	3.5	4.25
	2	1.6×10^5	2.5	2.8	3.33	4.75
	3	6.5×10^5	1.9	2.22	2.44	4

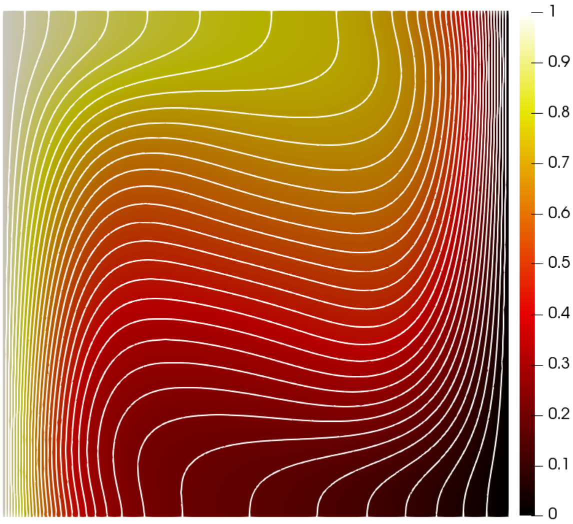
Table 5.7: Average number of Krylov iterations per Newton step as Ra increases for the constitutive relation - with $r = 1.6$ and $\mathrm{Di} = 0$, obtained using 7 multigrid cycles with 7 relaxation sweeps.



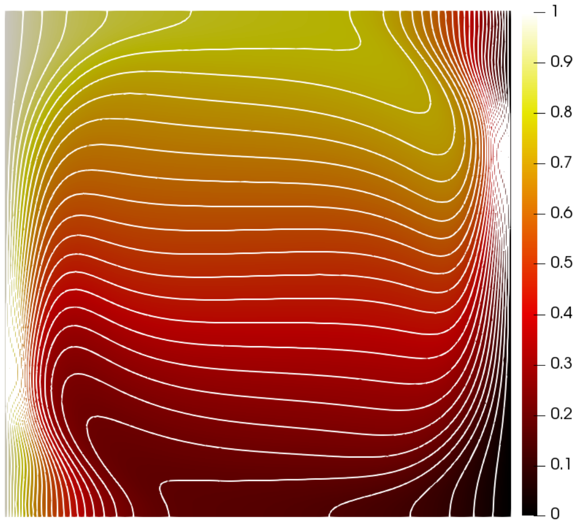
(a) Streamlines for $r = 2$.



(b) Streamlines for $r = 1.6$.



(c) Temperature contour for $r = 2$.



(d) Temperature contour for $r = 1.6$.

Figure 5.4: Streamlines and temperature contours for the heated cavity with the power-law constitutive relation (5.117) and $\mathrm{Ra} = 2 \times 10^4$.

5.3.2 Bingham flow in a cooling channel

Let $\Omega := (0, 40) \times (-1, 1)$, and consider the following boundary conditions for the temperature:

$$\nabla\theta \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \setminus (\Gamma_H \cup \Gamma_C), \quad \theta = \begin{cases} \theta_H, & \text{on } \Gamma_H, \\ 0, & \text{on } \Gamma_C, \end{cases}$$

where $\theta_H > 0$, and $\Gamma_H := \{(x_1, x_2)^\top \in \partial\Omega : x_1 \leq 10\}$ and $\Gamma_C := \{(x_1, x_2)^\top \in \partial\Omega : x_2 \in \{-1, 1\},\, 10 < x_1\}$. We employ in this section the Bingham constitutive relation that is obtained by setting $\sigma \equiv 0$ in (5.5). In this example we will consider a forced convection regime, in which the buoyancy effects are not taken into account, i.e. the steady counterpart of the system (5.1). The non-dimensional form of the system then reads:

$$-\operatorname{div}\mathbf{S} + \operatorname{Re}\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = 0 \qquad \text{in } \Omega, \tag{5.118a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{5.118b}$$

$$-\frac{1}{\operatorname{Pe}}\operatorname{div}(\nabla\theta) + \operatorname{div}(\boldsymbol{u}\theta) = \frac{\operatorname{Br}}{\operatorname{Pe}}\mathbf{S} : \mathbf{D}(\boldsymbol{u}) \qquad \text{in } \Omega, \tag{5.118c}$$

$$\sqrt{\varepsilon^2 + |\mathbf{D}(\boldsymbol{u})|^2}\mathbf{S} = (\operatorname{Bn}\hat{\tau}(\theta) + 2\hat{\mu}(\theta)|\mathbf{D}(\boldsymbol{u})|)\mathbf{D}(\boldsymbol{u}), \text{ in } \Omega, \tag{5.118d}$$

where $\varepsilon > 0$ is the regularisation parameter (c.f. (4.7a)), and the Reynolds, Péclet, Bingham and Brinkman numbers are defined as

$$\operatorname{Re} = \frac{\rho_* U R}{\nu_0}, \quad \operatorname{Pe} = \frac{\rho_* c_p U R}{\kappa_0}, \quad \operatorname{Bn} = \frac{\tau_0 R}{\nu_0 U}, \quad \operatorname{Br} = \frac{\nu_0 U^2}{\kappa_0 \theta_H}, \tag{5.119}$$

where R is the radius of the channel, U is the average velocity at the inlet and τ_0 is the value of the yield stress at the inlet. Two choices for the (non-dimensional) viscosity and yield stress are considered here:

$$\text{Problem (Q1): } \hat{\mu}(\theta) := a_1\theta + a_2 \quad \hat{\tau}(\theta) := 1.$$

$$\text{Problem (Q2): } \hat{\mu}(\theta) := 1 \quad \hat{\tau}(\theta) := b_1\theta + b_2.$$

The values of a_1 and a_2 are chosen so that the viscosity is unity at the inlet and increases by a factor of 20 at the outlet (which means that the effective Bingham number decreases by the same factor). The constants b_1 and b_2 are such that the Bingham number is 1.5 at the inlet, and 9 at the outlet when a temperature drop of 15 is applied. As for the velocity, we impose the following boundary conditions:

$$(\mathbf{S} - p\boldsymbol{I})(1, 0)^\top \cdot (1, 0)^\top = 0,\ \boldsymbol{u} \cdot (0, 1)^\top = 0 \text{ on } \Gamma_{\text{out}}, \quad \boldsymbol{u} = \boldsymbol{u}_B \text{ on } \Gamma_{\text{in}},$$

$$\boldsymbol{u} = \mathbf{0} \text{ on } \partial\Omega \setminus (\Gamma_{\text{in}} \cup \Gamma_{\text{out}}),$$

γ	# refs	# dofs	ε			
			1×10^{-3}	1×10^{-4}	2×10^{-5}	1×10^{-5}
10^3	1	2.7×10^4	12.8	22	51	48
	2	1.0×10^5	14.8	33.5	55	49
	3	4.3×10^5	13.5	17	25	17
	4	1.7×10^6	11.71	8.8	13	12
10^5	1	2.7×10^4	2.6	2	2	1.33
	2	1.0×10^5	2.6	2.25	1.4	1.15
	3	4.3×10^5	2	1.33	1.14	1
	4	1.7×10^6	1.75	1.33	1.15	1.07

Table 5.8: Average number of Krylov iterations per Newton step as ε decreases for Problem (Q1) with $k=2$, $\mathrm{Pe}=10$, $\theta_H=10$, $\mathrm{Br}=0.1$.

where $\Gamma_{\text{in}} := \{x_1 = 0\}$, $\Gamma_{\text{out}} := \{x_1 = 40\}$, and $\boldsymbol{u}_B$ is the fully developed Poiseuille flow for the isothermal problem with $\mathrm{Bn}=1.5$, for which the exact solution is available (see (4.41)). In order to obtain better initial guesses for Newton's method, secant continuation was employed: given two previously computed solutions $\boldsymbol{z}_1, \boldsymbol{z}_2$ corresponding to the parameters $\varepsilon_1, \varepsilon_2$, respectively, the initial guess for Newton's method at ε is chosen as

$$\frac{\varepsilon-\varepsilon_2}{\varepsilon_2-\varepsilon_1}(\boldsymbol{z}_2-\boldsymbol{z}_1)+\boldsymbol{z}_2.$$

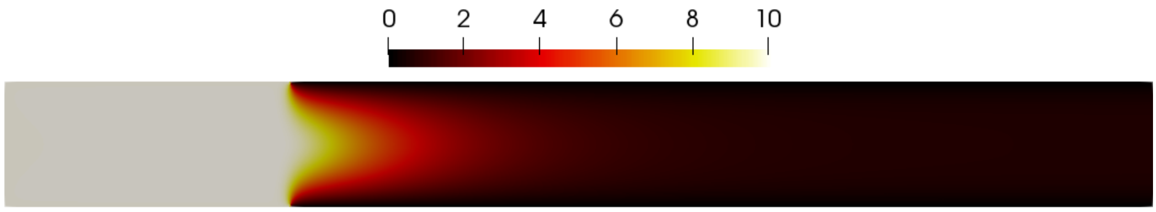
For this (arguably more complex) problem, the multigrid algorithm for the top block ceased to be effective. Tables 5.8–5.9 show the average number of Krylov iterations per Newton step obtained when using a sparse direct solver for the top block. It can be observed that for large values of the augmented Lagrangian parameter γ it is still possible to have an excellent control of the Schur complement. This suggests that it might be worthwhile to follow the same strategy of using a block preconditioner that singles out the pressure, while attempting a different strategy for constructing a scalable solver for the top block.

Figures 5.5–5.6 show the temperature field and the yielded/unyielded regions of the fluid. The results are qualitatively similar to those found in [VWA05], where an algorithm based on the augmented Lagrangian method was applied to a similar problem (neglecting the convective term and viscous dissipation). While it is known that a method based on regularisation, such as the one applied here, is not the most appropriate if one wishes to locate the exact position of the yield surfaces, it can still be useful to obtain the general features of the flow. For example, the solutions found

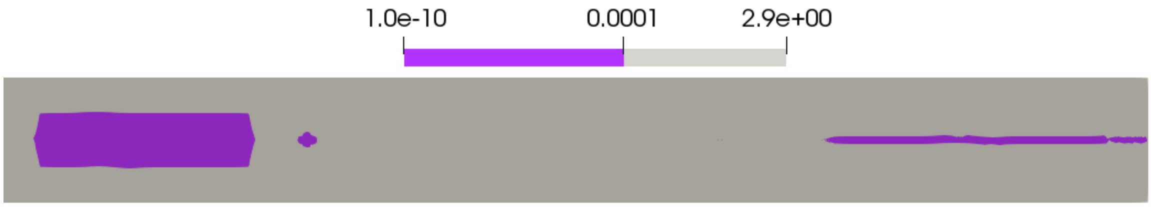
γ	# refs	# dofs	ε			
			1×10^{-3}	1×10^{-4}	5×10^{-5}	2×10^{-5}
10^3	1	2.7×10^4	12	17.5	26.6	*
	2	1.0×10^5	11.3	16.25	17	23
	3	4.3×10^5	12.88	13.67	14.5	13
	4	1.7×10^6	6.48	*	*	*
10^5	1	2.7×10^4	1.78	2.16	1.75	1.2
	2	1.0×10^5	1.6	1.3	1.16	1.07
	3	4.3×10^5	1.78	1.17	1.05	1
	4	1.7×10^6	1.31	1.13	1.03	1

Table 5.9: Average number of Krylov iterations per Newton step as ε decreases for Problem (Q2) with $k = 2$, $\mathrm{Pe} = 10$, $\theta_H = 10$, $\mathrm{Br} = 0$. The symbol * means that the maximum permitted number of nonlinear iterations was reached.

here show no unyielded regions in the transition zone where the temperature field varies with the mean flow direction, which is the expected behaviour [VWA05].

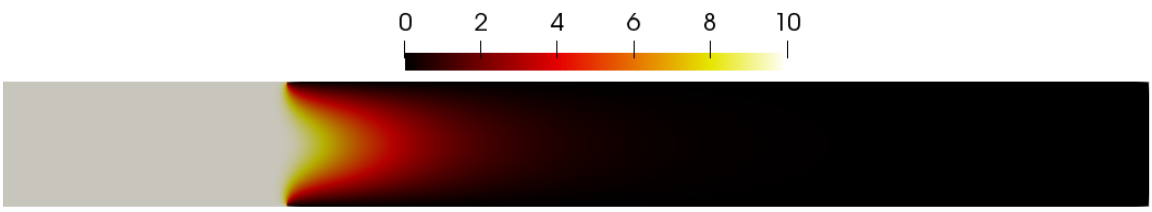


(a) Temperature.

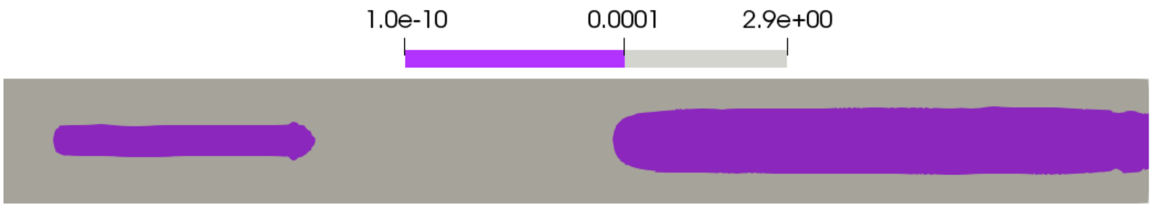


(b) Magnitude of the symmetric velocity gradient.

Figure 5.5: Temperature field and yielded regions for the Bingham flow on a cooling channel (Problem (Q1)), with $\mathrm{Pe} = 10$, $\theta_H = 10$, $\mathrm{Br} = 0.1$.



(a) Temperature.



(b) Magnitude of the symmetric velocity gradient.

Figure 5.6: Temperature field and yielded regions for Bingham flow on a cooling channel (Problem (Q2)), with $\mathrm{Pe} = 10$, $\theta_H = 10$, $\mathrm{Br} = 0$.

Chapter 6

Conclusions and Future Work

In this thesis we analysed several numerical schemes for the approximation of implicitly constituted incompressible flows, with a particular emphasis on formulations including the shear stress $\mathbf{S}$ as one of the variables.

The first part of the thesis focused on systems where the effects of the temperature were neglected. After introducing the notation and some technical tools in Chapter 2, we presented a finite element formulation for the unsteady system where the constitutive relation is given by a maximal monotone r-graph $\mathcal{A}$. Subsequently, inspired by the works [DKS13, ST19], (weak) convergence of the sequence of numerical approximations to a weak solution of the system was established. The numerical scheme involved three different levels of approximation: $k \to \infty$, $(n, m) \to \infty$, and $l \to \infty$, corresponding to a regularisation of the monotone graph, the time and space discretisation, and a penalty term, respectively. Crucial in the analysis were a proper choice of finite element spaces (e.g. the stress and velocity spaces must satisfy an appropriate inf-sup condition), and the application of a Lipschitz truncation argument, in order to identify the constitutive relation in the optimal range $r > \frac{2d}{d+2}$. Very importantly, numerical experiments showed that the formulation we studied exhibits the optimal rates of convergence in the particular cases where these are known (Section 3.2.1). In addition, numerical experiments also show that the penalty term seemingly provides no benefit in practice (Section 3.2.2).

A possibility for future research would be to extend the discrete Lipschitz truncation introduced in [DKS13] to the transient setting; this would allow one to carry out the approximation using a single limit, and to avoid the use of a penalty term. We also observed that in the simulations it is possible to use directly the constitutive relation in the form $\mathbf{G}(\mathbf{S}, \mathbf{D}) = \mathbf{0}$. It might be worth investigating in the future whether a different characterisation of the monotone graph $\mathcal{A}$ could allow one to go through the analysis without demanding the use of a selection.

Knowing that our discretisation was convergent, we set out in Chapter 4 to find an efficient way to perform the simulations. Following the work of Benzi and Olshanskii [BO06], and Farrell, Mitchell and Wechsung [FMW19, FMSW20a], we introduced an augmented Lagrangian preconditioner for the steady system, based on a formulation employing the Scott–Vogelius element for the velocity and pressure. Since the addition of the augmented Lagrangian term allows one to have an excellent control of the Schur complement, the main effort was devoted to the development of a monolithic multigrid solver for the stress-velocity block. Thanks to results by Schöberl [Sch99a, Sch99b], it is known that (for symmetric and positive definite systems) in order to obtain a robust multigrid operator, a robust relaxation scheme and robust transfer operators are essential. In this work this was achieved by performing a patchwise relaxation that captures the kernel of the divergence and by making sure that the transfer operators map divergence-free velocities to almost divergence-free velocities; the transfer operators for the stress were based on a supermesh projection. The resulting preconditioner showed remarkable robustness properties with respect with the rheological parameters. Despite this, there is still much to be understood; for instance, the current theory for robust subspace decompositions applies only to symmetric and positive systems, and it would be a worthwhile and challenging pursuit to extend it to the non-symmetric and indefinite case.

In Chapter 5 we extended the results from previous chapters to the anisothermal setting, where the constitutive relation is defined by the 2-graph parametrised by the temperature introduced in [MZ18]. For the unsteady system we considered an approximation scheme consisting of four different levels: $m \to \infty$, $n \to \infty$, $l \to \infty$, and $k \to \infty$, corresponding to the time discretisation and Galerkin limit for the temperature, the space discretisation for the rest of the variables, the quasi-compressible approximation, and a regularisation of the convective term and constitutive relation, respectively. The proof of this convergence result owes its technical nature to the fact that the viscous dissipation term $\mathbf{S} : \mathbf{D}(\boldsymbol{u})$ is included in the energy equation. We then considered a setting in which the convergence proof could be significantly simplified: the steady Oberbeck–Boussinesq approximation neglecting viscous dissipation; in that case the approximation scheme only requires the use of a single limiting procedure. The augmented Lagrangian preconditioner of Chapter 4 was then extended to this system, and performed well on the Navier–Stokes and power-law models, including the viscous heating, and temperature-dependent viscosity and heat conductivity. The convergence result is based on a localised Minty lemma that is tailored to a specific

2-graph, and it might be worth looking in the future for an extension to more general r-graphs.

The preconditioners presented in this thesis are computationally expensive, especially in 3D, and it will be the subject of future research to try to find ways to reduce the cost. A feasible alternative could be the use of discretisations based on $H(\mathrm{div})$–L^2-type elements for the velocity and pressure, since in that case the relaxation can employ smaller patches. The development of a preconditioner based on these ideas for discretisations of the transient problem involving high order time-stepping techniques could also have potentially a big impact in applications. We will now mention a few other possible extensions of the work contained in this thesis that will form the basis of future research.

Non-monotone constitutive relations

All the constitutive relations considered in this thesis were required to satisfy a monotonicity condition, because in this setting there are more tools available for the identification of nonlinear limits (e.g. Minty's lemma). However, in some applications it is possible to find material responses that follow an S-shape curve in the $|\mathbf{D}|$-$|\mathbf{S}|$ plane, and are thus non-monotone (see e.g. [BHMP97, DF04, FOG$^+$12]). If one follows the traditional approach of trying to employ a constitutive relation of the kind $\mathbf{S} = \boldsymbol{\mathcal{S}}(\mathbf{D})$, then the resulting function is necessarily multi-valued, which could lead to many technical complications. In contrast, the framework of implicitly constituted fluids can very naturally capture this behaviour.

An example of a constitutive relation that can model this type of response was introduced by Le Roux and Rajagopal in [LRR13], and is given by the expression

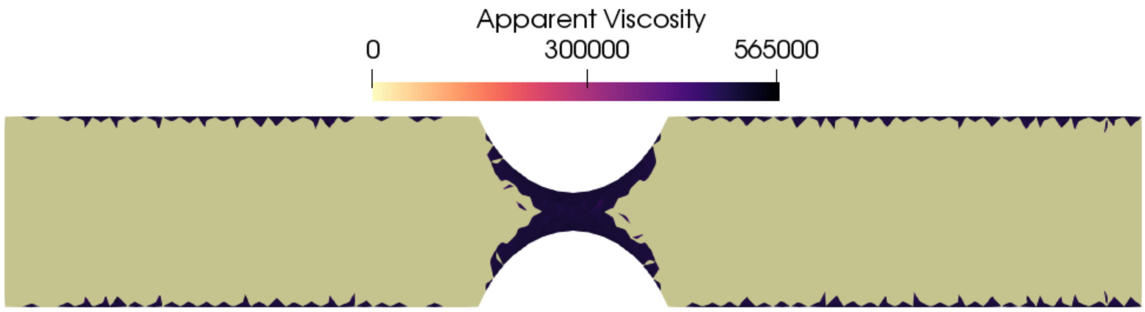
$$\mathbf{D} = \boldsymbol{\mathcal{D}}(\mathbf{S}) := \left[a(1 + b|\mathbf{S}|^2)^{\frac{q-2}{2}} + c\right]\mathbf{S}, \tag{6.1}$$

where a, b, c are positive constants and q is a real number; when $q < 1$ the relation (6.1) is in general non-monotone (see [LRR13] for details). An advantage of the relation (6.1) is that it can be shown to be thermodynamically consistent [JP18]. Although there could be multiple states of the shear stress $\mathbf{S}$ corresponding to just one value of the symmetric velocity gradient $\mathbf{D}$, potentially leading to some ambiguity, it seems that when solved as a transient problem no ambiguity arises and the evolution of the system is well defined, and can be observed to avoid the non-monotone sections of the constitutive relation.

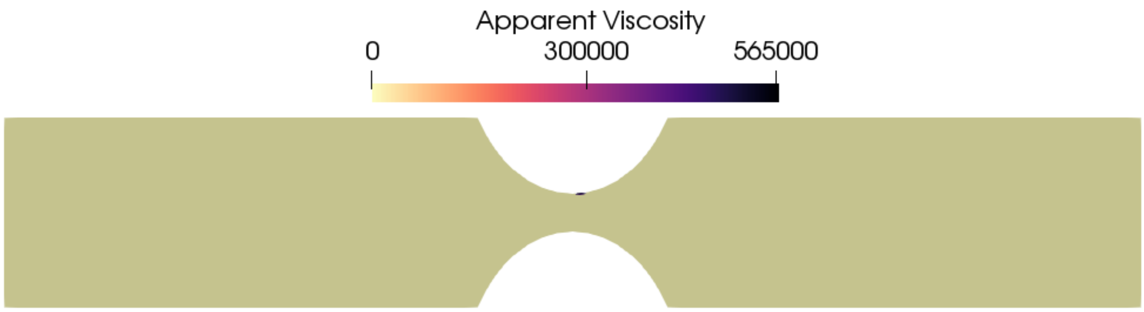
Even though the existence of weak solutions to the isothermal system supplemented with the constitutive relation (6.1) is still an open problem, a numerical scheme yielding solutions with the expected behaviour was proposed in [JMPT19]. This numerical scheme is based on a fixed point iteration of 3-field approximation where the unknowns are the velocity $\boldsymbol{u}$, the pressure p, and the apparent viscosity μ_a, which is defined as:

$$\mu_a(t, \boldsymbol{x}) := \frac{1}{2} \frac{|\mathbf{S}(t, \boldsymbol{x})|}{|\mathbf{D}(t, \boldsymbol{x})|}. \tag{6.2}$$

In the future we wish to investigate whether an $\mathbf{S}$-$\boldsymbol{u}$-p formulation, like the one analysed in Chapter 3, could provide better approximation properties; it seems especially well-suited for relations of the form (6.1), since the relation is already written in terms of the shear stress $\mathbf{S}$. Some preliminary computations using such a formulation seem promising: Figure 6.1 shows the apparent viscosity of a fluid in a channel with a narrowing, satisfying the relation (6.1), for two different values of the magnitude of the velocity at the inlet, after one time-step. In the problem with a larger inlet velocity, an interface is observed to appear, across which the apparent viscosity increases by five orders of magnitude. This is precisely the expected behaviour of non-monotone constitutive relations [JMPT19].



(a) Large inlet velocity.



(b) Small inlet velocity.

Figure 6.1: Apparent viscosity of a fluid on a channel with a narrowing, satisfying the Le Roux–Rajagopal constitutive relation, with $a = 1$, $b = 0.1$, $c = 10^{-6}$ and $q = 0.5$.

Convergence results via weak-strong uniqueness

The concept of weak solution used in the mathematical theory of fluid dynamics makes it possible to employ many tools of functional analysis to tackle the problem of existence of solutions, but makes the matter of uniqueness more challenging. For instance, the uniqueness of the Leray solutions of a Newtonian fluid is a notoriously difficult open problem (that of an implicitly constituted fluid even more so). In spite of that, one can obtain useful information from the so-called weak-strong uniqueness results, that guarantee that a weak and a strong solution, emanating from the same initial condition, will coincide as long as the latter exists. Weak-strong uniqueness results are useful in the analysis of singular limits and stability of stationary states [SR09, BFN18], for instance, and have been obtained in different contexts (see e.g. [FN12, DST12, LT13, AFN19a]). An example of such a result for the incompressible Navier–Stokes system was established by Prodi and Serrin [Pro59, Ser62].

As for incompressible implicitly constituted fluids, the only weak-strong uniqueness result available has been obtained by Abbatiello and Feireisl [AF20]. This result is striking in that weak-strong uniqueness is shown to hold even when using a rather weakened notion of solution: the solution is only required to be such that $\boldsymbol{u} \in C_w([0,T]; L^2_{\mathrm{div}}(\Omega)^d)$, $\mathbf{D}(\boldsymbol{u}) \in \mathcal{M}(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \cap L^\infty(0,T;W^{-1,2}(\Omega)^{d\times d})$, and $\mathbf{S} \in L^1_{\mathrm{sym}}(Q)^{d\times d}$ satisfy in the sense of distributions the following system, for any $\tau \in (0,T)$:

$$
\begin{gathered}
\partial_t \boldsymbol{u} + \mathrm{div}(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p = \mathrm{div}\,\mathbf{S} - \mathrm{div}\,\mathcal{R}_{\boldsymbol{u}},\\
\mathrm{div}\,\boldsymbol{u} = 0,\\
\frac{1}{2}\int_\Omega |\boldsymbol{u}(\tau,\cdot)|^2 + \int_\Omega \mathrm{d}\frac{1}{2}\,\mathrm{tr}[\mathcal{R}_{\boldsymbol{u}}](\tau) + \int_0^\tau\int_\Omega [F(\mathbf{D}(\boldsymbol{u})) + F^*(\mathbf{S})] \leq \frac{1}{2}\int_\Omega |\boldsymbol{u}_0|^2,
\end{gathered}
$$

where $\boldsymbol{u}_0 \in L^2_{\mathrm{div}}(\Omega)^d$ is the initial velocity, $\mathcal{R}_{\boldsymbol{u}} \in L^\infty(0,T;\mathcal{M}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ is the Reynolds stress capturing possible concentrations of the convective term, and $F\colon \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ is a convex function defining the constitutive relation via $\mathbf{S}:\mathbf{D}(\boldsymbol{u}) = F(\mathbf{D}(\boldsymbol{u})) + F^*(\mathbf{S})$. This is called a *dissipative weak solution*; note that this concept of solution does not require the constitutive relation to be satisfied in any pointwise sense.

If a sequence of numerical approximations can be shown to generate a dissipative weak solution (in particular the energy inequality must hold), then it will converge strongly to the (uniquely defined) strong solution, as long as it exists. The generality of the framework of implicitly constituted fluids gives a very small hope of obtaining convergence results through error estimates, and thus makes this approach more attractive. We note that this strategy has been applied to prove convergence of finite volume schemes to solutions of the compressible Navier–Stokes system [FLMMS19].

An important application of this approach would be a convergence result related to the transient anisothermal system studied in Chapter 5. The proof of Theorem 5.1.5 involves the use of a quasi-compressibility approximation and explicitly avoids the use of no-slip boundary conditions for the velocity, both of which involve restrictions that do not seem to be necessary when performing simulations. To our knowledge, there are no convergence results available anywhere in the literature that take into account the viscous dissipation term $\mathbf{S}:\mathbf{D}(\boldsymbol{u})$ and allow no-slip boundary conditions, and we wish to investigate whether the approach based on weak-strong uniqueness can fill this gap.

Flows with activation parameters

In the mathematical description of fluid mechanics it is very common to assume that fluids stick to solid walls. For internal flows (i.e. for which $\boldsymbol{u}\cdot\boldsymbol{n}=0$), this amounts to saying that the tangential component of the velocity $\boldsymbol{u}_\tau$ vanishes at the boundary, leading to the no-slip condition $\boldsymbol{u}|_{\partial\Omega}=\mathbf{0}$. However, in practice there are many indications that this is an approximation that only works well in certain regimes, and that in general some slip at the boundary takes place (see [HL03, Den01, Den04] and the references therein). The transition between these two types of responses can be conveniently described by means of the so-called threshold-slip (or stick-slip) boundary condition:

$$\begin{cases} -\boldsymbol{s} = \sigma_1 \frac{\boldsymbol{u}_\tau}{|\boldsymbol{u}_\tau|} + \gamma \boldsymbol{u}_\tau, & \text{if } |\boldsymbol{s}| > \sigma_1, \\ \boldsymbol{u}_\tau = 0, & \text{if } |\boldsymbol{s}| \leq \sigma_1, \end{cases} \qquad \text{on } \partial\Omega, \tag{6.3}$$

where $\boldsymbol{s} := (\mathbf{S}\boldsymbol{n})_\tau$ and $\sigma_1, \gamma \geq 0$. The relation (6.3) describes a fluid that sticks to the wall before activation, and slips after activation (if $\sigma_1 = 0$ it reduces to Navier's slip condition). The expression (6.3) can be naturally restated in terms of an implicit function

$$\boldsymbol{h}(\boldsymbol{s}, \boldsymbol{u}_\tau) = \mathbf{0} \qquad \text{on } \partial\Omega, \tag{6.4}$$

and is amenable to analysis with the techniques used to study implicitly constituted fluids. The first large data existence result for a system satisfying (6.4) was obtained in [BM17, BM16] (see also [BMR20]). Extensions that incorporate additional physical responses can be found in [BM19, MZ18]; in particular, the work [MZ18] allowed some temperature dependence by working with the boundary condition

$$\boldsymbol{h}(\boldsymbol{s}, \boldsymbol{u}_\tau, \theta) := \gamma(\theta)\frac{(|\boldsymbol{u}_\tau| - \sigma_2(\theta))^+}{|\boldsymbol{u}_\tau|}\boldsymbol{u}_\tau - \frac{(|\boldsymbol{s}| - \sigma_1(\theta))^+}{|\boldsymbol{s}|}\boldsymbol{s} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{6.5}$$

where $\gamma, \sigma_1, \sigma_2$ are continuous functions of the temperature θ, satisfying certain compatibility conditions.

Besides representing physical behaviour more accurately, activated boundary conditions of the kind (6.3) have the advantage that they allow one to obtain an integrable pressure, which can be crucial in the analysis (c.f. Theorem 5.1.5). The numerical analysis of systems subject to boundary conditions of the form (6.4) or (6.5) is still very underdeveloped and there are no convergence results to date; this is something that we would like to address in future work. In addition, on the computational side, it is not obvious what is the best way to implement such boundary conditions accurately and efficiently; e.g. it might be worth investigating whether an approach based on Nitsche's method can be advantageous.

In this thesis we encountered some examples of fluids with activation parameters, for which some kind of regularisation was applied in order to be able to use Newton's method (e.g. a Bercovier–Engelman-like regularisation for Bingham fluids in Section 5.3.2, and a Papanastasiou-like regularisation for activated Euler fluids in Section 3.2.3). While useful to obtain general qualitative features of the flow, and easier to implement, the regularisation approach is not very satisfactory when locating the exact position of the yield surfaces [PFM09, SW17]. A popular alternative to regularisation methods that deals better with this issue is the so-called augmented Lagrangian method [GW10, Sar10]. For a Bingham fluid (neglecting the convective term), the idea behind the method is to reformulate as a minimisation problem of the following non-smooth functional, defined for all divergence-free $\boldsymbol{v} \in H^1_0(\Omega)^d$:

$$I(\boldsymbol{v}) := \int_\Omega \nu|\mathbf{D}(\boldsymbol{v})|^2 + \tau\int_\Omega |\mathbf{D}(\boldsymbol{v})| - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}, \tag{6.6}$$

where $\nu > 0$ is the viscosity, $\tau \geq 0$ is the yield stress, $\boldsymbol{f} \in L^2(\Omega)^d$ is the body force. Introducing a new variable $\boldsymbol{\tau} := \mathbf{D}(\boldsymbol{v})$, and treating this definition as a constraint, then (after handling the divergence-free constraint with a Lagrange multiplier as well) one sees the minimisation problem has a saddle point formulation that can be written in terms of the Lagrangian

$$\begin{aligned}\mathcal{L}((\boldsymbol{v},\boldsymbol{\tau}),(p,\boldsymbol{\sigma})) := &\int_\Omega \nu|\boldsymbol{\tau}|^2 + \int_\Omega \tau|\boldsymbol{\tau}| - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} - \int_\Omega p \operatorname{div}\boldsymbol{v}\\ &+ \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : (\mathbf{D}(\boldsymbol{v}) - \boldsymbol{\tau}) + \frac{\alpha}{2}\int_\Omega |\mathbf{D}(\boldsymbol{v}) - \boldsymbol{\tau}|^2,\end{aligned}$$

where p and $\boldsymbol{\sigma}$ take the roles of the Lagrange multipliers, and $\alpha \geq 0$ is an augmentation parameter. The saddle point problem is then handled with an Uzawa-like iteration. In the future we would like to develop a similar algorithm for the numerical approximation of flow with activated Euler rheology and explore applications to the simulation of boundary layers.

One of the drawbacks of the augmented Lagrangian method is that convergence can be very slow; the search for ways to accelerate the algorithm is still a very active area of research, see e.g. [DMGT18, TMGP16]. At the same time, there have been other efforts to keep the potentially faster convergence of Newton's method when approximating the non-regularised problem, by slightly reformulating the system and applying a solver related to the regularised problem as a preconditioner [AHOV11, Sar16]. It might be worth investigating in the future whether ideas from the theory of augmented Lagrangian preconditioners and robust multigrid can be applied to make the aforementioned algorithms more efficient and scalable, e.g. by either applying them to the Newtonian solves occurring in every iteration of the algorithm described in [DMGT18], or by improving the preconditioner based on regularisation from [AHOV11, Sar16].

Additional physical responses

While the constitutive laws of the form $\mathbf{G}(\mathbf{S}, \mathbf{D}) = \mathbf{0}$ considered in this work capture a wide range of models, there are many kinds of physical responses that fall outside of this framework. We mention just a couple of examples.

All the constitutive relations studied in this thesis were *algebraic*, i.e. no derivatives of $\mathbf{S}$ or $\mathbf{D}$ were included. This excludes the hugely important family of constitutive relations that describe viscoelastic fluids, which as the name suggests, are fluids that can store energy in the form of strain energy (elastic behaviour), and at the same time dissipate energy (viscous behaviour); see.g. [BAH87] for an overview of several models that describe viscoelastic fluids. One of the most popular models used in the literature is the Oldroyd-B model, for which the constitutive relation can be written as:

$$\mathbf{S} + \frac{\mu_1}{G}\overset{\triangledown}{\mathbf{S}} = 2(\mu_1 + \mu_2)\mathbf{D} + 2\frac{\mu_1\mu_2}{G}\overset{\triangledown}{\mathbf{D}}, \tag{6.7}$$

where $G > 0$, $\mu_1, \mu_2 \geq 0$, and the symbol $\overset{\triangledown}{\boldsymbol{\tau}}$ denotes the upper convected Oldroyd derivative of $\boldsymbol{\tau}$:

$$\overset{\triangledown}{\boldsymbol{\tau}} := \frac{\partial \boldsymbol{\tau}}{\partial \mathrm{t}} + \boldsymbol{u} \cdot \nabla \boldsymbol{\tau} - (\nabla \boldsymbol{u})\,\boldsymbol{\tau} - \boldsymbol{\tau}\,(\nabla \boldsymbol{u})^{\top}.$$

Note that when $\mu_1 = 0$, the constitutive relation (6.7) reduces to the Newtonian model.

The analysis of models such as (6.7) is very challenging because of the hyperbolic character of the constitutive relation, and the mathematical and numerical analysis of such systems is still a very active area of research. In recent years, it has become more clear that the addition of a stress diffusion term $\Delta \mathbf{S}$ is physically justified and can make the analysis more tractable, see e.g. [BBM20, MPSS18, BB11, EKL89]. A very interesting research direction for the future would be the analysis of numerical schemes and the development of preconditioners for systems that take these new developments into account, while possibly incorporating the non-Newtonian phenomena captured by the implicit constitutive relations studied in this thesis (e.g. elastoviscoplastic incorporate in addition a yield-stress [Sar07, CSD+11]).

Another example of fluids with many industrial applications are the so-called electrorheological fluids, which are suspensions whose material properties change drastically when exposed to electric fields [Růž04, HL10]. The constitutive relations in models describing such fluids therefore have to depend on the electric field E. A popular form for the constitutive relation describing electrorheological fluids is given by [RR01]

$$\mathbf{S} = \boldsymbol{\mathcal{S}}(\mathbf{D}, E) := \mu(E)(\kappa + |\mathbf{D}|^2)^{\frac{p(E)-2}{2}}\mathbf{D}, \tag{6.8}$$

where $\kappa > 0$, and μ, p are functions of the electric field; in some cases an E-dependent yield stress is also included [RPYO03]. As for the numerical analysis of electrorheological fluids, we are only aware of the works [BBD16, CHP10, Die02]. The fact that the power-law exponent p is not constant anymore is one of the main difficulties that arises in the analysis of such systems. A similar difficulty appears in the study of chemically reacting non-Newtonian fluids, in which the power-law exponent depends on the concentration c of a chemical, which is a function solving some reaction-diffusion equation [BP14, KS18, KPS18]. These topics present ample opportunity for contributions, especially from the point of view of the development of preconditioning and fast solvers.

These examples are by no means exhaustive, there are many other types of models for which we think an extension of our work on numerical analysis and preconditioning techniques would be interesting and valuable, such as implicitly constituted compressible fluids [AFN19b, FKN20], fluids with pressure-dependent viscosity and/or yield-stress [BMR09, HLS12, JFP06], models taking thixotropy into account [GW10, WVF09], and some models of turbulence [Lew94, CFC97, BM19], just to name a few.

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